4.3 Permutations With Some Identical Items

Often, you will deal with permutations in which some items are identical.

INVESTIGATE & INQUIRE: What Is in a Name?

1. In their mathematics class, John and Jenn calculate the number of permutations of all the letters of their first names.
   a) How many permutations do you think John finds?
   b) List all the permutations of John's name.
   c) How many permutations do you think Jenn finds?
   d) List all the permutations of Jenn's name.
   e) Why do you think there are different numbers of permutations for the two names?
2. a) List all the permutations of the letters in your first name. Is the number of permutations different from what you would calculate using the ${}_nP_n = n!$ formula? If so, explain why.
   b) List and count all the permutations of a word that has two identical pairs of letters. Compare your results with those your classmates found with other words. What effect do the identical letters have on the number of different permutations?
   c) Predict how many permutations you could make with the letters in the word *googol*. Work with several classmates to verify your prediction by writing out and counting all of the possible permutations.
3. Suggest a general formula for the number of permutations of a word that has two or more identical letters.

As the investigation above suggests, you can develop a general formula for permutations in which some items are identical.

Example 1 Permutations With Some Identical Elements

Compare the different permutations for the words *DOLE*, *DOLL*, and *LOLL*.

Solution

The following are all the permutations of *DOLE*:

DOLE	DOEL	DLOE	DLEO	DEOL	DELO
ODLE	ODEL	OLDE	OLED	OEDL	OELD
LODE	LOED	LDOE	LDEO	LEOD	LEDO
EOLD	EODL	ELOD	ELDO	EDOL	EDLO

There are 24 permutations of the four letters in *DOLE*. This number matches what you would calculate using ${}_4P_4 = 4!$

To keep track of the permutations of the letters in the word *DOLL*, use a subscript to distinguish the one *L* from the other.

$\mathrm{DOLL_1}$	$\mathrm{DOL_1L}$	$\mathrm{DLO\ L_1}$	$\mathrm{DL\ L_1O}$	$\mathrm{DL_1OL}$	$\mathrm{DL_1LO}$
$\mathrm{ODLL_1}$	$\mathrm{ODL_1L}$	$\mathrm{OLDL_1}$	$\mathrm{OLL_1D}$	$\mathrm{OL_1DL}$	$\mathrm{OL_1LD}$
$\mathrm{LODL_1}$	$\mathrm{LOL_1D}$	$\mathrm{LDOL_1}$	$\mathrm{LDL_1O}$	$\mathrm{LL_1OD}$	$\mathrm{LL_1DO}$
$\mathrm{L_1OLD}$	$\mathrm{L_1ODL}$	$\mathrm{L_1LOD}$	$\mathrm{L_1LDO}$	$\mathrm{L_1DOL}$	$\mathrm{L_1DLO}$

Of the 24 arrangements listed here, only 12 are actually different from each other. Since the two *L*s are in fact identical, each of the permutations shown in black is duplicated by one of the permutations shown in red. If the two *L*s in a permutation trade places, the resulting permutation is the same as the original one. The two *L*s can trade places in ${}_2P_2 = 2!$ ways.

Thus, the number of different arrangements is

$$\frac{4!}{2!} = \frac{24}{2}$$
$$= 12$$

In other words, to find the number of permutations, you divide the total number of arrangements by the number of ways in which you can arrange the identical letters. For the letters in *DOLL*, there are four ways to choose the first letter, three ways to choose the second, two ways to choose the third, and one way to choose the fourth. You then divide by the 2! or 2 ways that you can arrange the two *L*s.

Similarly, you can use subscripts to distinguish the three *L*s in *LOLL*, and then highlight the duplicate arrangements.

$\mathrm{L_2OLL_1}$	$\mathrm{L_2OL_1L}$	$\mathrm{L_2LOL_1}$	$\mathrm{L_2LL_1O}$	$\mathrm{L_2L_1OL}$	$\mathrm{L_2L_1LO}$
$\mathrm{OL_2LL_1}$	$\mathrm{OL_2L_1L}$	$\mathrm{OLL_2L_1}$	$\mathrm{OLL_1L_2}$	$\mathrm{OL_1L_2L}$	$\mathrm{OL_1LL_2}$
$\mathrm{LOL_2L_1}$	$\mathrm{LOL_1L_2}$	$\mathrm{LL_2OL_1}$	$\mathrm{LL_2L_1O}$	$\mathrm{LL_1OL_2}$	$\mathrm{LL_1L_2O}$
$\mathrm{L_1OLL_2}$	$\mathrm{L_1OL_2L}$	$\mathrm{L_1LOL_2}$	$\mathrm{L_1LL_2O}$	$\mathrm{L_1L_2OL}$	$\mathrm{L_1L_2LO}$

The arrangements shown in black are the only different ones. As with the other two words, there are 24 possible arrangements if you distinguish between the identical *L*s. Here, the three identical *L*s can trade places in $_3P_3 = 3!$ ways.

Thus, the number of permutations is $\frac{4!}{3!} = 4$.

You can generalize the argument in Example 1 to show that the number of permutations of a set of n items of which a are identical is $\frac{n!}{a!}$.

Example 2 Tile Patterns

Tanisha is laying out tiles for the edge of a mosaic. How many patterns can she make if she uses four yellow tiles and one each of blue, green, red, and grey tiles?

Solution

Here, $n = 8$ and $a = 4$.

$$\begin{aligned}\frac{8!}{4!} &= 8 \times 7 \times 6 \times 5 \\ &= 1680\end{aligned}$$

Tanisha can make 1680 different patterns with the eight tiles.

Example 3 Permutation With Several Sets of Identical Elements

The word *bookkeeper* is unusual in that it has three consecutive double letters. How many permutations are there of the letters in *bookkeeper*?

Solution

If each letter were different, there would be 10! permutations, but there are two *o*s, two *k*s, and three *e*s. You must divide by 2! twice to allow for the duplication of the *o*s and *k*s, and then divide by 3! to allow for the three *e*s:

$$\begin{aligned}\frac{10!}{2!2!3!} &= \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4}{2 \times 2} \\ &= 151\ 200\end{aligned}$$

There are 151 200 permutations of the letters in *bookkeeper*.

The number of permutations of a set of n objects containing a identical objects of one kind, b identical objects of a second kind, c identical objects of a third kind, and so on is $\frac{n!}{a!b!c!\ldots}$.

Example 4 Applying the Formula for Several Sets of Identical Elements

Barbara is hanging a display of clothing imprinted with the school's crest on a line on a wall in the cafeteria. She has five sweatshirts, three T-shirts, and four pairs of sweatpants. In how many ways can Barbara arrange the display?

Solution

Here, $a = 5$, $b = 3$, $c = 4$, and the total number of items is 12.

So,

$$\frac{n!}{a!b!c!} = \frac{12!}{5!3!4!}$$
$$= 27\ 720$$

Barbara can arrange the display in 27 720 different ways.

Project Prep

The game you design for your probability project could involve permutations of identical objects.

Key Concepts

- When dealing with permutations of n items that include a identical items of one type, b identical items of another type, and so on, you can use the formula $\frac{n!}{a!b!c!\ldots}$.

Communicate Your Understanding

1. Explain why there are fewer permutations of a given number of items if some of the items are identical.

2. **a)** Explain why the formula for the numbers of permutations when some items are identical has the denominator $a!b!c!\ldots$ instead of $a \times b \times c\ldots$.

b) Will there ever be cases where this denominator is larger than the numerator? Explain.

c) Will there ever be a case where the formula does not give a whole number answer? What can you conclude about the denominator and the numerator? Explain your reasoning.

Practise

A

1. Identify the indistinguishable items in each situation.
   a) The letters of the word *mathematics* are arranged.
   b) Dina has six notebooks, two green and four white.
   c) The cafeteria prepares 50 chicken sandwiches, 100 hamburgers, and 70 plates of French fries.
   d) Thomas and Richard, identical twins, are sitting with Marianna and Megan.

2. How many permutations are there of all the letters in each name?
   a) Inverary   b) Beamsville
   c) Mattawa   d) Penetanguishene

3. How many different five-digit numbers can be formed using three 2s and two 5s?

4. How many different six-digit numbers are possible using the following numbers?
   a) 1, 2, 3, 4, 5, 6   b) 1, 1, 1, 2, 3, 4
   c) 1, 3, 3, 4, 4, 5   d) 6, 6, 6, 6, 7, 8

Apply, Solve, Communicate

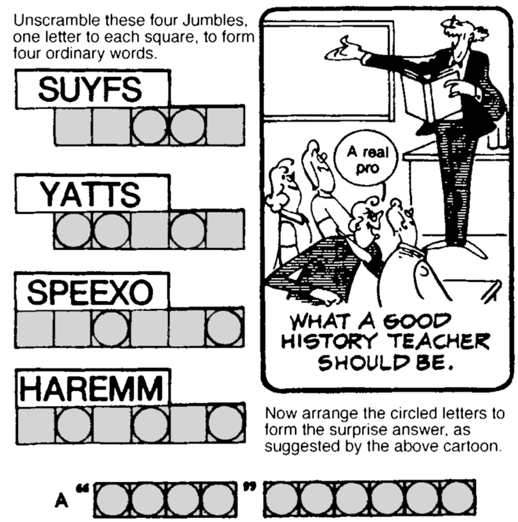
B

5. **Communication** A coin is tossed eight times. In how many different orders could five heads and three tails occur? Explain your reasoning.

6. **Inquiry/Problem Solving** How many 7-digit even numbers less than 3 000 000 can be formed using all the digits 1, 2, 2, 3, 5, 5, 6?

7. Kathryn's soccer team played a good season, finishing with 16 wins, 3 losses, and 1 tie. In how many orders could these results have happened? Explain your reasoning.

8. a) Calculate the number of permutations for each of the jumbled words in this puzzle.
   b) Estimate how long it would take to solve this puzzle by systematically writing out the permutations.



WEB CONNECTION

www.mcgrawhill.ca/links/MDM12

For more word jumbles and other puzzles, visit the above web site and follow the links. Find or generate two puzzles for a classmate to solve.

9. **Application** Roberta is a pilot for a small airline. If she flies to Sudbury three times, Timmins twice, and Thunder Bay five times before returning home, how many different itineraries could she follow? Explain your reasoning.

10. After their training run, six members of a track team split a bag of assorted doughnuts. How many ways can the team share the doughnuts if the bag contains
    a) six different doughnuts?
    b) three each of two varieties?
    c) two each of three varieties?

11. As a project for the photography class, Haseeb wants to create a linear collage of photos of his friends. He creates a template with 20 spaces in a row. If Haseeb has 5 identical photos of each of 4 friends, in how many ways can he make his collage?

12. **Communication** A used car lot has four green flags, three red flags, and two blue flags in a bin. In how many ways can the owner arrange these flags on a wire stretched across the lot? Explain your reasoning.

13. **Application** Malik wants to skateboard over to visit his friend Gord who lives six blocks away. Gord's house is two blocks west and four blocks north of Malik's house. Each time Malik goes over, he likes to take a different route. How many different routes are there for Malik if he only travels west or north?

ACHIEVEMENT CHECK

Knowledge / Understanding	Thinking / Inquiry / Problem Solving	Communication	Application

14. Fran is working on a word puzzle and is looking for four-letter "scrambles" from the clue word *calculate*.

a) How many of the possible four-letter scrambles contain four different letters?

b) How many contain two *a*s and one other pair of identical letters?

c) How many scrambles consist of any two pairs of identical letters?

d) What possibilities have you not yet taken into account? Find the number of scrambles for each of these cases.

e) What is the total number of four-letter scrambles taking all cases into account?

15. Chapter Problem Ten students have been nominated for the positions of secretary, treasurer, social convenor, and fundraising chair. In how many ways can these positions be filled if the Norman twins are running and plan to switch positions on occasion for fun since no one can tell them apart?

16. **Inquiry / Problem Solving** In how many ways can all the letters of the word *CANADA* be arranged if the consonants must always be in the order in which they occur in the word itself?

C

17. Glen works part time stocking shelves in a grocery store. The manager asks him to make a pyramid display using 72 cans of corn, 36 cans of peas, and 57 cans of carrots. Assume all the cans are the same size and shape. On his break, Glen tries to work out how many different ways he could arrange the cans into a pyramid shape with a triangular base.

a) Write a formula for the number of different ways Glen could stack the cans in the pyramid.

b) Estimate how long it will take Glen to calculate this number of permutations by hand.

c) Use computer software or a calculator to complete the calculation.

18. How many different ways are there of arranging seven green and eight brown bottles in a row, so that exactly one pair of green bottles is side-by-side?

19. In how many ways could a class of 18 students divide into groups of 3 students each?