

Pascal's Triangle

The array of numbers shown below is called Pascal's triangle in honour of French mathematician, Blaise Pascal (1623–1662). Although it is believed that the 14th century Chinese mathematician Chu Shi-kie knew of this array and some of its applications, Pascal discovered it independently at age 13. Pascal found many mathematical uses for the array, especially in probability theory.

Pascal's method for building his triangle is a simple iterative process similar to those described in, section 1.1. In Pascal's triangle, each term is equal to the sum of the two terms immediately above it. The first and last terms in each row are both equal to 1 since the only term immediately above them is also always a 1.

If $t_{n,r}$ represents the term in row n , position r , then $t_{n,r} = t_{n-1,r-1} + t_{n-1,r}$.

For example, $t_{6,2} = t_{5,1} + t_{5,2}$. Note that both the row and position labelling begin with 0.

			1				Row 0
		1	1				Row 1
	1	2	1				Row 2
	1	3	3	1			Row 3
	1	4	6	4	1		Row 4
	1	5	10	10	5	1	Row 5
1	6	15	20	15	6	1	Row 6



Chu Shi-kie's triangle

				$t_{0,0}$			
			$t_{1,0}$	$t_{1,1}$			
		$t_{2,0}$	$t_{2,1}$	$t_{2,2}$			
		$t_{3,0}$	$t_{3,1}$	$t_{3,2}$	$t_{3,3}$		
	$t_{4,0}$	$t_{4,1}$	$t_{4,2}$	$t_{4,3}$	$t_{4,4}$		
	$t_{5,0}$	$t_{5,1}$	$t_{5,2}$	$t_{5,3}$	$t_{5,4}$	$t_{5,5}$	
$t_{6,0}$	$t_{6,1}$	$t_{6,2}$	$t_{6,3}$	$t_{6,4}$	$t_{6,5}$	$t_{6,6}$	

WEB CONNECTION

www.mcgrawhill.ca/links/MDM12

Visit the above web site and follow the links to learn more about Pascal's triangle. Write a brief report about an application or an aspect of Pascal's triangle that interests you.


INVESTIGATE & INQUIRE: Row Sums

- Find the sums of the numbers in each of the first six rows of Pascal's triangle and list these sums in a table.
- Predict the sum of the entries in
 - row 7
 - row 8
 - row 9
- Verify your predictions by calculating the sums of the numbers in rows 7, 8, and 9.
- Predict the sum of the entries in row n of Pascal's triangle.
- List any other patterns you find in Pascal's triangle. Compare your list with those of your classmates. Do their lists suggest further patterns you could look for?

In his book *Mathematical Carnival*, Martin Gardner describes Pascal's triangle as "so simple that a 10-year old can write it down, yet it contains such inexhaustible riches and links with so many seemingly unrelated aspects of mathematics, that it is surely one of the most elegant of number arrays."

Example 1 Pascal's Method

- The first six terms in row 25 of Pascal's triangle are 1, 25, 300, 2300, 12 650, and 53 130. Determine the first six terms in row 26.
- Use Pascal's method to write a formula for each of the following terms:
 - $t_{12,5}$
 - $t_{40,32}$
 - $t_{n+1,r+1}$

Solution

- $$t_{26,0} = 1$$

$$t_{26,1} = 1 + 25 = 26$$

$$t_{26,2} = 25 + 300 = 325$$

$$t_{26,3} = 300 + 2300 = 2600$$

$$t_{26,4} = 2300 + 12\,650 = 14\,950$$

$$t_{26,5} = 12\,650 + 53\,130 = 65\,780$$
- $t_{12,5} = t_{11,4} + t_{11,5}$
 - $t_{40,32} = t_{39,31} + t_{39,32}$
 - $t_{n+1,r+1} = t_{n,r} + t_{n,r+1}$

Example 2 Row Sums

Which row in Pascal's triangle has the sum of its terms equal to 32 768?

Solution

From the investigation on page 248, you know that the sum of the terms in any row n is 2^n . Dividing 32 768 by 2 repeatedly, you find that $32\,768 = 2^{15}$. Thus, it is row 15 of Pascal's triangle that has terms totalling 32 768.

Example 3 Divisibility

Determine whether $t_{n,2}$ is divisible by $t_{n,1}$ in each row of Pascal's triangle.

Solution

Row	$\frac{t_{n,2}}{t_{n,1}}$	Divisible?
0 and 1	n/a	n/a
2	0.5	no
3	1	yes
4	1.5	no
5	2	yes
6	2.5	no
7	3	yes

It appears that $t_{n,2}$ is divisible by $t_{n,1}$ only in odd-numbered rows. However, $2t_{n,2}$ is divisible by $t_{n,1}$ in all rows that have three or more terms.

Example 4 Triangular Numbers

Coins can be arranged in the shape of an equilateral triangle as shown.

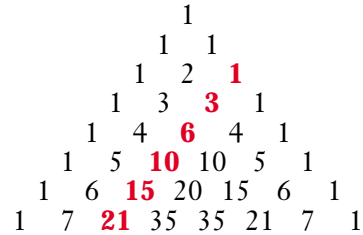


- Continue the pattern to determine the numbers of coins in triangles with four, five, and six rows.
- Locate these numbers in Pascal's triangle.
- Relate Pascal's triangle to the number of coins in a triangle with n rows.
- How many coins are in a triangle with 12 rows?

Solution

- a) The numbers of coins in the triangles follow the pattern $1 + 2 + 3 + \dots$ as shown in the table below.
- b) The numbers of coins in the triangles match the entries on the third diagonal of Pascal's triangle.

Number of Rows	Number of Coins	Term in Pascal's Triangle
1	1	$t_{2,2}$
2	3	$t_{3,2}$
3	6	$t_{4,2}$
4	10	$t_{5,2}$
5	15	$t_{6,2}$
6	21	$t_{7,2}$



- c) Compare the entries in the first and third columns of the table. The row number of the term from Pascal's triangle is always one greater than the number of rows in the equilateral triangle. The position of the term in the row, r , is always 2. Thus, the number of coins in a triangle with n rows is equal to the term $t_{n+1,2}$ in Pascal's triangle.

d) $t_{12+1,2} = t_{13,2}$
 $= 78$

A triangle with 12 rows contains 78 coins.

Numbers that correspond to the number of items stacked in a triangular array are known as **triangular numbers**. Notice that the n th triangular number is also the sum of the first n positive integers.

Example 5 Perfect Squares

Can you find a relationship between perfect squares and the sums of pairs of entries in Pascal's triangle?

Solution

Again, look at the third diagonal in Pascal's triangle.

n	n^2	Entries in Pascal's Triangle	Terms in Pascal's Triangle
1	1	1	$t_{2,2}$
2	4	$1 + 3$	$t_{2,2} + t_{3,2}$
3	9	$3 + 6$	$t_{3,2} + t_{4,2}$
4	16	$6 + 10$	$t_{4,2} + t_{5,2}$

Each perfect square greater than 1 is equal to the sum of a pair of adjacent terms on the third diagonal of Pascal's triangle: $n^2 = t_{n,2} + t_{n+1,2}$ for $n > 1$.

Key Concepts

- Each term in Pascal's triangle is equal to the sum of the two adjacent terms in the row immediately above: $t_{n,r} = t_{n-1,r-1} + t_{n-1,r}$ where $t_{n,r}$ represents the r th term in row n .
- The sum of the terms in row n of Pascal's triangle is 2^n .
- The terms in the third diagonal of Pascal's triangle are triangular numbers. Many other number patterns occur in Pascal's triangle.

Communicate Your Understanding

1. Describe the symmetry in Pascal's triangle.
2. Explain why the triangular numbers in Example 4 occur in Pascal's triangle.

Practise

A

1. For future use, make a diagram of the first 12 rows of Pascal's triangle.
2. Express as a single term from Pascal's triangle.
 - a) $t_{7,2} + t_{7,3}$
 - b) $t_{51,40} + t_{51,41}$
 - c) $t_{18,12} - t_{17,12}$
 - d) $t_{n,r} - t_{n-1,r}$
3. Determine the sum of the terms in each of these rows in Pascal's triangle.
 - a) row 12
 - b) row 20
 - c) row 25
 - d) row $(n - 1)$
4. Determine the row number for each of the following row sums from Pascal's triangle.
 - a) 256
 - b) 2048
 - c) 16 384
 - d) 65 536

Apply, Solve, Communicate

B

5. Inquiry/Problem Solving

- a) Alternately add and subtract the terms in each of the first seven rows of Pascal's triangle and list the results in a table similar to the one below.

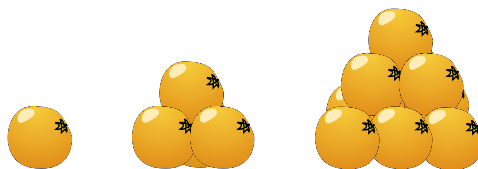
Row	Sum/Difference	Result
0	1	1
1	$1 - 1$	0
2	$1 - 2 + 1$	0
3	$1 - 3 + 3 - 1$	0
\vdots		

- b) Predict the result of alternately adding and subtracting the entries in the eighth row. Verify your prediction.
 - c) Predict the result for the n th row.
6. a) Predict the sum of the squares of the terms in the n th row of Pascal's triangle.
b) Predict the result of alternately adding and subtracting the squares of the terms in the n th row of Pascal's triangle.

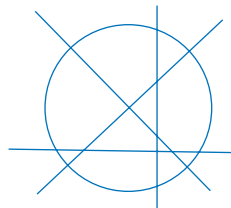
7. Communication

- a) Compare the first four powers of 11 with entries in Pascal's triangle. Describe any pattern you notice.
 - b) Explain how you could express row 5 as a power of 11 by regrouping the entries.
 - c) Demonstrate how to express rows 6 and 7 as powers of 11 using the regrouping method from part b). Describe your method clearly.
8. a) How many diagonals are there in
- i) a quadrilateral?
 - ii) a pentagon?
 - iii) a hexagon?
- b) Find a relationship between entries in Pascal's triangle and the maximum number of diagonals in an n -sided polygon.
- c) Use part b) to predict how many diagonals are in a heptagon and an octagon. Verify your prediction by drawing these polygons and counting the number of possible diagonals in each.
9. Make a conjecture about the divisibility of the terms in prime-numbered rows of Pascal's triangle. Confirm that your conjecture is valid up to row 11.
10. a) Which rows of Pascal's triangle contain only odd numbers? Is there a pattern to these rows?
- b) Are there any rows that have only even numbers?
- c) Are there more even or odd entries in Pascal's triangle? Explain how you arrived at your answer.

11. **Application** Oranges can be piled in a tetrahedral shape as shown. The first pile contains one orange, the second contains four oranges, the third contains ten oranges, and so on. The numbers of items in such stacks are known as **tetrahedral numbers**.



- a) Relate the number of oranges in the n th pile to entries in Pascal's triangle.
- b) What is the 12th tetrahedral number?
12. a) Relate the sum of the squares of the first n positive integers to entries in Pascal's triangle.
- b) Use part a) to predict the sum of the squares of the first ten positive integers. Verify your prediction by adding the numbers.
13. **Inquiry/Problem Solving** A straight line drawn through a circle divides it into two regions.
- a) Determine the maximum number of regions formed by n straight lines drawn through a circle. Use Pascal's triangle to help develop a formula.





15. The Fibonacci sequence is 1, 1, 2, 3, 5, 8, 13, 21, Each term is the sum of the previous two terms. Find a relationship between the Fibonacci sequence and the following version of Pascal's triangle.

```

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
...

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16. **Application** Toothpicks are laid out to form triangles as shown below. The first triangle contains 3 toothpicks, the second contains 9 toothpicks, the third contains 18 toothpicks, and so on.



- a) Relate the number of toothpicks in the n th triangle to entries in Pascal's triangle.
- b) How many toothpicks would the 10th triangle contain?
17. Design a 3-dimensional version of Pascal's triangle. Use your own criteria for the layers. The base may be any regular geometric shape, but each successive layer must have larger dimensions than the one above it.

18. a) Write the first 20 rows of Pascal's triangle on a sheet of graph paper, placing each entry in a separate square.
- b) Shade in all the squares containing numbers divisible by 2.
- c) Describe, in detail, the patterns produced.
- d) Repeat this process for entries divisible by other whole numbers. Observe the resulting patterns and make a conjecture about the divisibility of the terms in Pascal's triangle by various whole numbers.

19. **Communication**

- a) Describe the iterative process used to generate the terms in the triangle below.

$$\begin{array}{ccccccc}
 & & & & & & \frac{1}{1} \\
 & & & & & & \frac{1}{1} \\
 & & & & & & \frac{1}{2} & \frac{1}{2} \\
 & & & & & & \frac{1}{3} & \frac{1}{6} & \frac{1}{3} \\
 & & & & & & \frac{1}{4} & \frac{1}{12} & \frac{1}{12} & \frac{1}{4} \\
 & & & & & & \frac{1}{5} & \frac{1}{20} & \frac{1}{30} & \frac{1}{20} & \frac{1}{5} \\
 & & & & & & \frac{1}{6} & \frac{1}{30} & \frac{1}{60} & \frac{1}{60} & \frac{1}{30} & \frac{1}{6}
 \end{array}$$

- b) Write the entries for the next two rows.
- c) Describe three patterns in this triangle.
- d) Research why this triangle is called the harmonic triangle. Briefly explain the origin of the name, listing your source(s).