The Binomial Theorem

Pascal's Triangle and the Binomial Theorem

$$(x+y)^0 = 1$$

$$(x+y)^1 = 1x + 1y$$

$$(x+y)^2 = 1x^2 + 2xy + 1y^2$$

$$(x+y)^3 = 1x^3 + 3x^2y + 3xy^2 + 1y^3$$

$$(x+y)^4 = 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4$$

$$(x+y)^5 = 1x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + 1y^5$$

$$(x+y)^6 = 1x^6 + 6x^5y^1 + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + 1y^6$$

The Binomial Theorem

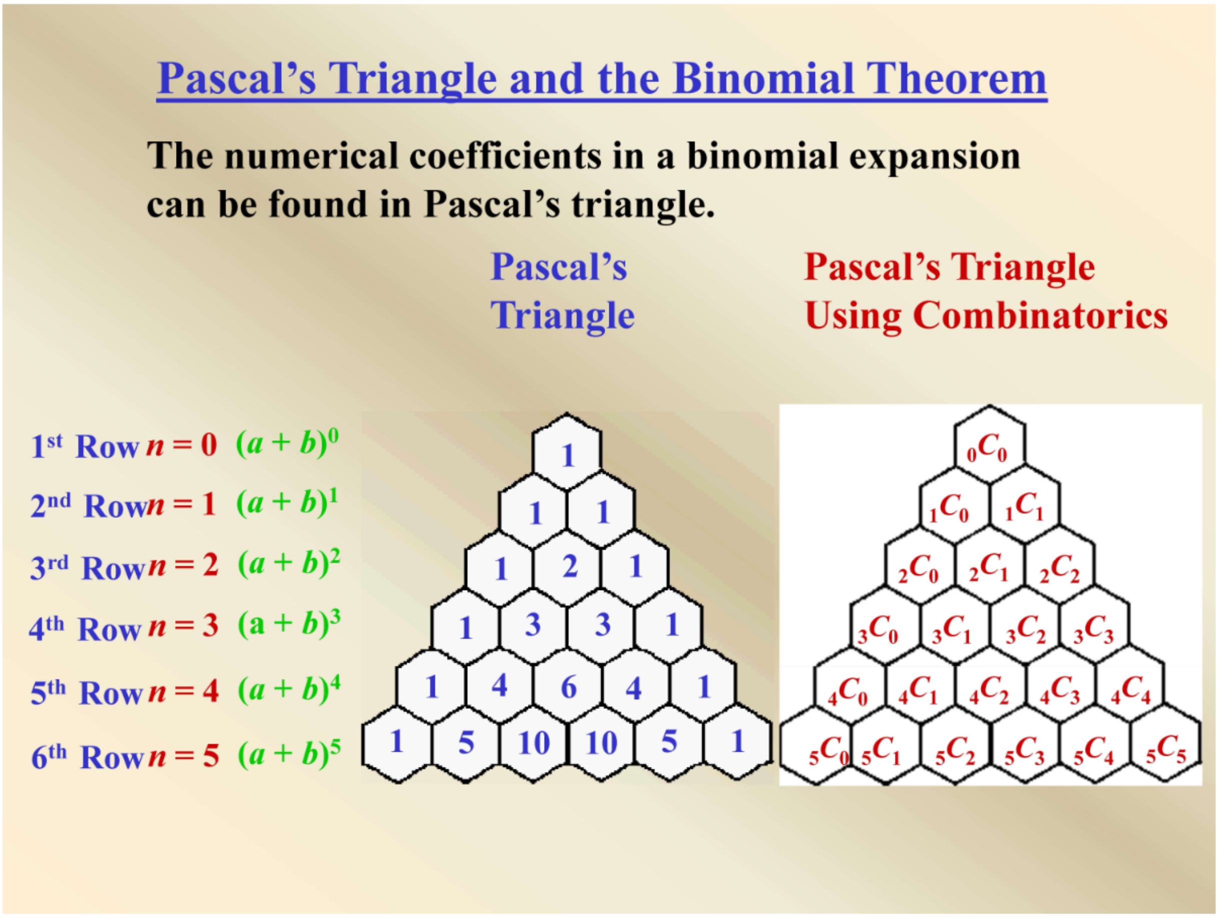
The **Binomial Theorem** is a formula used for expanding powers of binomials.

$$(a + b)^3 = (a + b)(a + b)(a + b)$$
$$= a^3 + 3a^2b + 3ab^2 + b^3$$

Each term of the answer is the product of three first-degree factors. For each term of the answer, an a and/or b is taken from each first-degree factor.

- The first term has no b. It is like choosing no b from three b's. The combination ${}_3C_0$ is the coefficient of the first term.
- The second term has one b. It is like choosing one b from three b's. The combination ${}_3C_1$ is the coefficient of the second term.
- The third term has two b's. It is like choosing two b's from three b's. The combination ${}_3C_2$ is the coefficient of the ~~first~~ third term.
- The fourth term has three b's. It is like choosing three b's from three b's. The combination ${}_3C_3$ is the coefficient of the third term.

$$(a + b)^3 = {}_3C_0\,a^3 + {}_3C_1\,a^2b + {}_3C_2\,ab^2 + {}_3C_3\,b^3$$


Pascal's Triangle and the Binomial Theorem
The numerical coefficients in a binomial expansion can be found in Pascal's triangle.
Pascal's Triangle
Pascal's Triangle Using Combinatorics
1st Row $n = 0$ $(a + b)^0$
2nd Row $n = 1$ $(a + b)^1$
3rd Row $n = 2$ $(a + b)^2$
4th Row $n = 3$ $(a + b)^3$
5th Row $n = 4$ $(a + b)^4$
6th Row $n = 5$ $(a + b)^5$
1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
${}_0C_0$
${}_1C_0$ ${}_1C_1$
${}_2C_0$ ${}_2C_1$ ${}_2C_2$
${}_3C_0$ ${}_3C_1$ ${}_3C_2$ ${}_3C_3$
${}_4C_0$ ${}_4C_1$ ${}_4C_2$ ${}_4C_3$ ${}_4C_4$
${}_5C_0$ ${}_5C_1$ ${}_5C_2$ ${}_5C_3$ ${}_5C_4$ ${}_5C_5$

Binomial Expansion - the General Term

$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

The degree of each term is 3.
For the variable a, the degree descends from 3 to 0.
For the variable b, the degree ascends from 0 to 3.

$(a + b)^3 = {}_3C_0\, a^{3-0}b^0 + {}_3C_1\, a^{3-1}b^1 + {}_3C_2\, a^{3-2}b^2 + {}_3C_3\, a^{3-3}b^3$

$(a + b)^n = {}_nC_0\, a^{n-0}b^0 + {}_nC_1\, a^{n-1}b^1 + {}_nC_2\, a^{n-2}b^2 + \ldots + {}_nC_k\, a^{n-k}b^k$

The general term is the $(k + 1)^{\text{th}}$ term: 9th $k=8$

$\boxed{t_{k+1} = {}_nC_k\, a^{n-k}\, b^k}$

$t_{8+1} = {}_{12}C_8\, a^{12-8}b^8$

$(a+b)^{12}$

Binomial Expansion - Practice

Expand the following.

$n = 4$
$a = 3x$
$b = 2$

a) $(3x + 2)^4$

$= {}_4C_0(3x)^4(2)^0 + {}_4C_1(3x)^3(2)^1 + {}_4C_2(3x)^2(2)^2 + {}_4C_3(3x)^1(2)^3 + {}_4C_4(3x)^0(2)^4$

$= 1(81x^4) + 4(27x^3)(2) + 6(9x^2)(4) + 4(3x)(8) + 1(16)$

$= 81x^4 + 216x^3 + 216x^2 + 96x + 16$

$n = 4$
$a = 2x$
$b = -3y$

b) $(2x - 3y)^4$

$= {}_4C_0(2x)^4(-3y)^0 + {}_4C_1(2x)^3(-3y)^1 + {}_4C_2(2x)^2(-3y)^2 + {}_4C_3(2x)^1(-3y)^3 + {}_4C_4(2x)^0(-3y)^4$

$= 1(16x^4) + 4(8x^3)(-3y) + 6(4x^2)(9y^2) + 4(2x)(-27y^3) + 81y^4$

$= 16x^4 - 96x^3y + 216x^2y^2 - 216xy^3 + 81y^4$

$$(2x-3y)^4$$

$$= {}_4C_0(2x)^4(-3y)^0 + {}_4C_1(2x)^3(-3y)^1 + {}_4C_2(2x)^2(-3y)^2 +$$

$$ {}_4C_3(2x)^1(-3y)^3 + {}_4C_4(2x)^0(-3y)^4$$

$$= (1)(16x^4) + 4(8x^3)(-3y) + 6(4x^2)(9y^2) +$$

$$(4)(2x)(-27y^3) + (1)(1)(81y^4)$$

$$= 16x^4 - 96x^3y + 216x^2y^2 - 216xy^3 + 81y^4$$

$n = 4$
$a = 2x$
$b = -3y$

b) $(2x - 3y)^4$

$$= {}_4C_0(2x)^4(-3y)^0 + {}_4C_1(2x)^3(-3y)^1 + {}_4C_2(2x)^2(-3y)^2 + {}_4C_3(2x)^1(-3y)^3 + {}_4C_4(2x)^0(-3y)^4$$

$$= 1(16x^4) + 4(8x^3)(-3y) + 6(4x^2)(9y^2) + 4(2x)(-27y^3) + 81y^4$$

$$= 16x^4 - 96x^3y + 216x^2y^2 - 216xy^3 + 81y^4$$

Binomial Expansion - Practice

$x^{-1} = \frac{1}{x}$

c) $\left(2x - \frac{3}{x}\right)^5$

$(2x - 3x^{-1})^5 = {}_5C_0(2x)^5(-3x^{-1})^0 + {}_5C_1(2x)^4(-3x^{-1})^1 + {}_5C_2(2x)^3(-3x^{-1})^2$

$+ {}_5C_3(2x)^2(-3x^{-1})^3 + {}_5C_4(2x)^1(-3x^{-1})^4 + {}_5C_5(2x)^0(-3x^{-1})^5$

$n = 5$
$a = 2x$
$b = -3x^{-1}$

$= 1(32x^5) + 5(16x^4)(-3x^{-1}) + 10(8x^3)(9x^{-2}) + 10(4x^2)(-27x^{-3})$

-243

$+ 5(2x)(81x^{-4}) + 1(-81x^{-5})$

$-243x^{-5}$

$= 32x^5 - 240x^3 + 720x^1 - 1080x^{-1} + 810x^{-3}\ \cancel{81x^{-5}}$

Finding a Particular Term in a Binomial Expansion

a) Find the eighth term in the expansion of $(3x - 2)^{11}$.

$n = 11$
$a = 3x$
$b = -2$
$k = 7$

$$t_{k+1} = {}_nC_k\, a^{n-k} b^k \rightarrow t_8 = ?$$

$$t_{7+1} = {}_{11}C_7\, (3x)^{11-7} (-2)^7$$
$$t_8 = {}_{11}C_7\, (3x)^4 (-2)^7$$
$$= 330(81x^4)(-128)$$
$$= -3\,421\,440\, x^4$$

$(a+b)^2 = O + O + O$

4 5^{th} 4

one term more than $\boxed{n}$

b) Find the middle term of $(a^2 - 3b^3)^8$.

$n = 8$
$a = a^2$
$b = -3b^3$
$k = 4$

$$t_{k+1} = {}_nC_k\, a^{n-k} b^k$$
$$t_{4+1} = {}_8C_4\, (a^2)^{8-4} (-3b^3)^4$$
$$t_5 = {}_8C_4\, (a^2)^4 (-3b^3)^4$$
$$= 70a^8(81b^{12})$$
$$= 5670a^8b^{12}$$

$n = 8$, therefore, there are nine terms. The fifth term is the middle term.

t_5

Finding a Particular Term in a Binomial Expansion

8th = k+1 → k=7

a) Find the eighth term in the expansion of $(\underbrace{3x}_{a}\underbrace{-2}_{b})^{11}$. n

$n = 11$
$a = 3x$
$b = -2$
$k = 7$ ✓

$$\boxed{t_{k+1} = {}_nC_k\, a^{n-k} b^k}$$

8th → k+1 k = 7

$$t_{7+1} = {}_{11}C_7\,(3x)^{11-7}(-2)^7$$
$$t_8 = {}_{11}C_7\,(3x)^4(-2)^7$$
$$= 330(81x^4)(-128)$$
$$= -3\,421\,440\,x^4$$

Total # of terms = n+1 = 9 terms

b) Find the middle term of $(a^2 - 3b^3)^8$.

$n = 8$ ✓
$a = a^2$
$b = -3b^3$
$k = 4$ ✓

$$t_{k+1} = {}_nC_k\, a^{n-k} b^k$$
$$t_{4+1} = {}_8C_4\,(a^2)^{8-4}(-3b^3)^4$$
$$t_5 = {}_8C_4\,(a^2)^4(-3b^3)^4$$
$$= 70a^8(81b^{12})$$
$$= 5670a^8b^{12}$$

$n = 8$, therefore, there are nine terms. The fifth term is the middle term.

5th = k+1

k = 4

Finding a Particular Term in a Binomial Expansion

with no variable (only the number)

Find the constant term of the expansion of $\left(2x-\frac{1}{x^2}\right)^{18}$.

$x^m \cdot x^n = x^{m+n}$

$n = 18$
$a = 2x$
$b = -x^{-2}$
$k = ?$

$$t_{k+1} = {}_{n}C_k\, a^{n-k} b^k$$

$$t_{k+1} = {}_{18}C_k\, (2x)^{18-k} (-x^{-2})^k$$

$$t_{k+1} = {}_{18}C_k\, (2^{18-k})(x^{18-k})(-1)^k(x^{-2k})$$

$$t_{k+1} = {}_{18}C_k\, 2^{18-k} (-1)^k \boxed{x^{18-k} x^{-2k}}$$

$$\boxed{t_{k+1} = {}_{18}C_k\, 2^{18-k} (-1)^k x^{18-3k}}$$

$(18-k)+(-2k)$
$= 18-3k$

general formula for k+1 th term specific to $\left(2x-\frac{1}{x^2}\right)^{18}$

$x^{18-3k} = x^0$

$18 - 3k = 0$

$-3k = -18$

$k = 6$

Substitute $k = 6$:

$$t_{6+1} = {}_{18}C_6\, 2^{18-6} (-1)^6 x^{18-3(6)}$$

$$t_7 = {}_{18}C_6 (2^{12})(-1)^6$$

$$t_7 = 76\,038\,144$$

Therefore, the constant term is 76 038 144.

Finding a Particular Term in a Binomial Expansion

$n=10$, $k=?$, $a=x^2$, $b=-x^{-1}$

Find the numerical coefficient of the x^{11} term of the expansion of $\left(x^2 - \dfrac{1}{x}\right)^{10}$.

k+1 th term

$(-1 \cdot x^{-1})^k = (-1)^k x^{-k}$

$n = 10$
$a = x^2$
$b = -x^{-1}$
$k = ?$

$t_{k+1} = {}_nC_k\, a^{n-k} b^k$

$t_{k+1} = {}_{10}C_k\, (x^2)^{10-k} (-x^{-1})^k$

$t_{k+1} = {}_{10}C_k\, x^{20-2k} (-1)^k x^{-k}$

$20-2k+(-k)$

$t_{k+1} = {}_{10}C_k (-1)^k \boxed{x^{20-2k} x^{-k}}$ x

$t_{k+1} = {}_{10}C_k\, (-1)^k \boxed{x^{20-3k}} = \boxed{x^{11}}$

$x^{20-3k} = x^{11}$

$20 - 3k = 11$

$-3k = -9$

$k = 3$

Substitute $k = 3$:

$t_{3+1} = {}_{10}C_3\, (-1)^3\, x^{20-3(3)}$

$t_4 = {}_{10}C_3\, (-1)^3 x^{11}$

$t_4 = -120\, x^{11}$

Therefore, the numerical coefficient of the x^{11} term is -120.

$(x+y)^9 = x^9 + \dots x^8y + \dots . x^5y^4$

Finding a Particular Term of the Binomial

4th term

One term in the expansion of $(2x - m)^7$ is $-15\ 120x^4y^3$. Find m.

$$\boxed{\begin{array}{l} n = 7 \\ a = 2x \\ b = -m \\ k = 3 \end{array}}$$

4-1=3

$$t_{k+1} = {}_nC_k\, a^{n-k}\, b^k$$

$$t_{k+1} = {}_7C_3\, (2x)^4\, (-m)^3$$

$$t_{k+1} = (35)\,(16x^4)\,(-m)^3$$

$$-15\ 120x^4y^3 = (560x^4)\,(-m)^3$$

$$\frac{-15\ 120x^4y^3}{560x^4} = -m^3$$

$$-27y^3 = -m^3$$

$$27y^3 = m^3$$

$$\sqrt[3]{27y^3} = m$$

$$3y = m$$

Therefore, m is $3y$.

Factoring Using The Binomial Theorem

a^n b^n

Rewrite $1+10x^2+40x^4+80x^6+80x^8+32x^{10}$ **as** $(a+b)^n$

\# of terms = 6 $n=6-1=5$

***n* = # of terms = 5**

First term = a^n **= 1 , so** $a=1$

Last term = b^n $=32x^{10}=(2x^2)^5$ **, so** $b=2x^2$

$$1+10x^2+40x^4+80x^6+80x^8+32x^{10}=(1+2x^2)^5$$