4.1 Organized Counting

The techniques and mathematical logic for counting possible arrangements or outcomes are useful for a wide variety of applications. A computer programmer writing software for a game or industrial process would use such techniques, as would a coach planning a starting line-up, a conference manager arranging a schedule of seminars, or a school board trying to make the most efficient use of its buses.

Combinatorics is the branch of mathematics dealing with ideas and methods for counting, especially in complex situations. These techniques are also valuable for probability calculations, as you will learn in Chapter 6.

INVESTIGATE & INQUIRE: Licence Plates

Until 1997, most licence plates for passenger cars in Ontario had three numbers followed by three letters. Suppose the provincial government had wanted all the vehicles registered in Ontario to have plates with the letters O, N, and T.

1. Draw a diagram to illustrate all the possibilities for arranging these three letters assuming that the letters can be repeated. How many possibilities are there?
2. How could you calculate the number of possible three-letter groups without listing them all?
3. Predict how many three-letter groups the letters O, N, T, and G can form.
4. How many three-letter groups do you think there would be if you had a choice of five letters?
5. Suggest a general strategy for counting all the different possibilities in situations like those above.

When you have to make a series of choices, you can usually determine the total number of possibilities without actually counting each one individually.

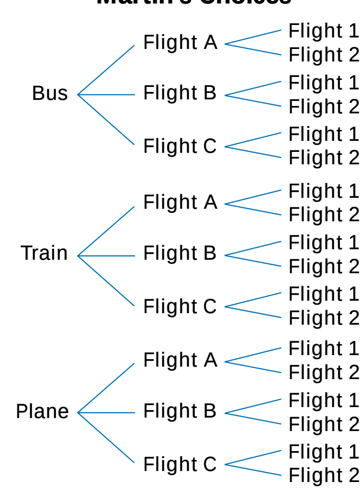
Example 1 Travel Itineraries

Martin lives in Kingston and is planning a trip to Vienna, Austria. He checks a web site offering inexpensive airfares and finds that if he travels through London, England, the fare is much lower. There are three flights available from Toronto to London and two flights from London to Vienna. If Martin can take a bus, plane, or train from Kingston to Toronto, how many ways can he travel from Kingston to Vienna?

Solution

You can use a tree diagram to illustrate and count Martin's choices. This diagram suggests another way to determine the number of options Martin has for his trip.

Martin's Choices



Choices for the first portion of trip:	3
Choices for the second portion of trip:	3
Choices for the third portion of trip:	2
Total number of choices:	$3 \times 3 \times 2 = 18$

In all, Martin has 18 ways to travel from Kingston to Vienna.

Example 2 Stereo Systems

Javon is looking at stereos in an electronics store. The store has five types of receivers, four types of CD players, and five types of speakers. How many different choices of stereo systems does this store offer?

Solution

For each choice of receiver, Javon could choose any one of the CD players. Thus, there are $5 \times 4 = 20$ possible combinations of receivers and CD players. For each of these combinations, Javon could then choose one of the five kinds of speakers.

The store offers a total of $5 \times 4 \times 5 = 100$ different stereo systems.

These types of counting problems illustrate the **fundamental** or **multiplicative counting principle**:

If a task or process is made up of stages with separate choices, the total number of choices is $m \times n \times p \times \ldots$, where m is the number of choices for the first stage, n is the number of choices for the second stage, p is the number of choices for the third stage, and so on.

Example 3 Applying the Fundamental Counting Principle

A school band often performs at benefits and other functions outside the school, so its members are looking into buying band uniforms. The band committee is considering four different white shirts, dress pants in grey, navy, or black, and black or grey vests with the school crest. How many different designs for the band uniform is the committee considering?

Project Prep

You can use the fundamental or multiplicative counting principle to help design the game for your probability project.

Solution

First stage: choices for the white shirts, $m = 4$
Second stage: choices for the dress pants, $n = 3$
Third stage: choices for the vests, $p = 2$
The total number of possibilities is

$$\begin{aligned} m \times n \times p &= 4 \times 3 \times 2 \\ &= 24 \end{aligned}$$

The band committee is considering 24 different possible uniforms.

In some situations, an **indirect method** makes a calculation easier.

Example 4 Indirect Method

Leora, a triathlete, has four pairs of running shoes loose in her gym bag. In how many ways can she pull out two unmatched shoes one after the other?

Solution

You can find the number of ways of picking unmatched shoes by subtracting the number of ways of picking matching ones from the total number of ways of picking any two shoes.

There are eight possibilities when Leora pulls out the first shoe, but only seven when she pulls out the second shoe. By the fundamental counting principle, the number of ways Leora can pick any two shoes out of the bag is $8 \times 7 = 56$. She could pick each of the matched pairs in two ways: left shoe then right shoe or right shoe then left shoe. Thus, there are $4 \times 2 = 8$ ways of picking a matched pair.

Leora can pull out two unmatched shoes in $56 - 8 = 48$ ways.

Sometimes you will have to count several subsets of possibilities separately.

Example 5 Signal Flags

Sailing ships used to send messages with signal flags flown from their masts. How many different signals are possible with a set of four distinct flags if a minimum of two flags is used for each signal?

Solution

A ship could fly two, three, or four signal flags.

Signals with two flags:	$4 \times 3 = 12$
Signals with three flags:	$4 \times 3 \times 2 = 24$
Signals with four flags:	$4 \times 3 \times 2 \times 1 = 24$
Total number of signals:	$12 + 24 + 24 = 60$

Thus, the total number of signals possible with these flags is 60.

In Example 5, you were counting actions that could not occur at the same time. When counting such **mutually exclusive** actions, you can apply the **additive counting principle** or **rule of sum**:

If one mutually exclusive action can occur in m ways, a second in n ways, a third in p ways, and so on, then there are $m + n + p \ldots$ ways in which one of these actions can occur.

Key Concepts

- Tree diagrams are a useful tool for organized counting.
- If you can choose from m items of one type and n items of another, there are $m \times n$ ways to choose one item of each type (fundamental or multiplicative counting principle).
- If you can choose from either m items of one type or n items of another type, then the total number of ways you can choose an item is $m + n$ (additive counting principle).
- Both the multiplicative and the additive counting principles also apply to choices of three or more types of items.
- Sometimes an indirect method provides an easier way to solve a problem.

Communicate Your Understanding

1. Explain the fundamental counting principle in your own words and give an example of how you could apply it.
2. Are there situations where the fundamental counting principle does not apply? If so, give one example.
3. Can you always use a tree diagram for organized counting? Explain your reasoning.

Practise

A

1. Construct a tree diagram to illustrate the possible contents of a sandwich made from white or brown bread, ham, chicken, or beef, and mustard or mayonnaise. How many different sandwiches are possible?
2. In how many ways can you roll either a sum of 4 or a sum of 11 with a pair of dice?
3. In how many ways can you draw a 6 or a face card from a deck of 52 playing cards?
4. How many ways are there to draw a 10 or a queen from the 24 cards in a euchre deck, which has four 10s and four queens?
5. Use tree diagrams to answer the following:
   a) How many different soccer uniforms are possible if there is a choice of two types of shirts, three types of shorts, and two types of socks?
   b) How many different three-scoop cones can be made from vanilla, chocolate, and strawberry ice cream?
   c) Suppose that a college program has six elective courses, three on English literature and three on the other arts. If the college requires students to take one of the English courses and one of the other arts courses, how many pairs of courses will satisfy these requirements?

Apply, Solve, Communicate

6. Ten different books and four different pens are sitting on a table. One of each is selected. Should you use the rule of sum or the product rule to count the number of possible selections? Explain your reasoning.

B

7. **Application** A grade 9 student may build a timetable by selecting one course for each period, with no duplication of courses. Period 1 must be science, geography, or physical education. Period 2 must be art, music, French, or business. Periods 3 and 4 must each be mathematics or English.
   a) Construct a tree diagram to illustrate the choices for a student's timetable.
   b) How many different timetables could a student choose?
8. A standard die is rolled five times. How many different outcomes are possible?
9. A car manufacturer offers three kinds of upholstery material in five different colours for this year's model. How many upholstery options would a buyer have? Explain your reasoning.
10. **Communication** In how many ways can a student answer a true-false test that has six questions. Explain your reasoning.

11. The final score of a soccer game is 6 to 3. How many different scores were possible at half-time?

12. A large room has a bank of five windows. Each window is either open or closed. How many different arrangements of open and closed windows are there?

13. **Application** A Canadian postal code uses six characters. The first, third, and fifth are letters, while the second, fourth, and sixth are digits. A U.S.A. zip code contains five characters, all digits.

  a) How many codes are possible for each country?

  b) How many more possible codes does the one country have than the other?

14. When three-digit area codes were introduced in 1947, the first digit had to be a number from 2 to 9 and the middle digit had to be either 1 or 0. How many area codes were possible under this system?

15. Asha builds new homes and offers her customers a choice of brick, aluminium siding, or wood for the exterior, cedar or asphalt shingles for the roof, and radiators or forced-air for the heating system. How many different configurations is Asha offering?

16. a) In how many ways could you choose two fives, one after the other, from a deck of cards?

  b) In how many ways could you choose a red five and a spade, one after the other?

  c) In how many ways could you choose a red five or a spade?

  d) In how many ways could you choose a red five or a heart?

  e) Explain which counting principles you could apply in parts a) to d).

17. Chapter Problem Ten students have been nominated for a students' council executive. Five of the nominees are from grade 12, three are from grade 11, and the other two are from grades 9 and 10.

  a) In how many ways could the nominees fill the positions of president and vice-president if all ten are eligible for these senior positions?

  b) How many ways are there to fill these positions if only grade 11 and grade 12 students are eligible?

18. **Communication**

  a) How many different licence plates could be made using three numbers followed by three letters?

  b) In 1997, Ontario began issuing licence plates with four letters followed by three numbers. How many different plates are possible with this new system?

  c) Research the licence plate formats used in the other provinces. Compare and contrast these formats briefly and suggest reasons for any differences between the formats.

19. In how many ways can you arrange the letters of the word *think* so that the *t* and the *h* are separated by at least one other letter?

20. **Application** Before the invention of the telephone, Samuel Morse (1791–1872) developed an efficient system for sending messages as a series of dots and dashes (short or long pulses). International code, a modified version of Morse code, is still widely used.

  a) How many different characters can the international code represent with one to four pulses?

  b) How many pulses would be necessary to represent the 72 letters of the Cambodian alphabet using a system like Morse code?

ACHIEVEMENT CHECK

Knowledge / Understanding	Thinking / Inquiry / Problem Solving	Communication	Application

21. Ten finalists are competing in a race at the Canada Games.

a) In how many different orders can the competitors finish the race?

b) How many ways could the gold, silver, and bronze medals be awarded?

c) One of the finalists is a friend from your home town. How many of the possible finishes would include your friend winning a medal?

d) How many possible finishes would leave your friend out of the medal standings?

e) Suppose one of the competitors is injured and cannot finish the race. How does that affect your previous answers?

f) How would the competitor's injury affect your friend's chances of winning a medal? Explain your reasoning. What assumptions have you made?

C

22. A locksmith has ten types of blanks for keys. Each blank has five different cutting positions and three different cutting depths at each position, except the first position, which only has two depths. How many different keys are possible with these blanks?

23. **Communication** How many 5-digit numbers are there that include the digit 5 and exclude the digit 8? Explain your solution.

24. **Inquiry / Problem Solving** Your school is purchasing a new type of combination lock for the student lockers. These locks have 40 positions on their dials and use a three-number combination.

a) How many combinations are possible if consecutive numbers cannot be the same?

b) Are there any assumptions that you have made? Explain.

c) Assuming that the first number must be dialled clockwise from 0, how many different combinations are possible?

d) Suppose the first number can also be dialled counterclockwise from 0. Explain the effect this change has on the number of possible combinations.

e) If you need four numbers to open the lock, how many different combinations are possible?

25. **Inquiry / Problem Solving** In chess, a knight can move either two squares horizontally plus one vertically or two squares vertically plus one horizontally.

a) If a knight starts from one corner of a standard 8×8 chessboard, how many different squares could it reach after

i) one move?

ii) two moves?

iii) three moves?

b) Could you use the fundamental counting principle to calculate the answers for part a)? Why or why not?