4.2

Factorials and Permutations

In many situations, you need to determine the number of different orders in which you can choose or place a set of items.

INVESTIGATE & INQUIRE: Numbers of Arrangements

Consider how many different ways a president and a vice-president could be chosen from eight members of a students' council.

1. a) Have one person in your class make two signs, writing *President* on one and *Vice-President* on the other. Now, choose two people to stand at the front of the class. Using the signs to indicate which person holds each position, decide in how many ways you can choose a president and a vice-president from the two people at the front of the class.

   b) Choose three students to be at the front of the class. Again using the signs to indicate who holds each position, determine how many ways you can choose a president and a vice-president from the three people at the front of the class.

   c) Repeat the process with four students. Do you see a pattern in the number of ways a president and a vice-president can be chosen from the different sizes of groups? If so, what is the pattern? If not, continue the process with five students and then with six students.

   d) When you see a pattern, predict how many ways a president and a vice-president can be chosen from the eight members of the students' council.

   e) Suggest other ways of simulating the selection of a president and a vice-president for the students' council.

2. Suppose that each of the eight members of the students' council has to give a brief speech at an assembly. Consider how you could determine the number of different orders in which they could speak.
   a) Choose two students from your class and list all the possible orders in which they could speak.
   b) Choose three students and list all the possible orders in which they could speak.
   c) Repeat this process with four students.
   d) Is there an easy method to organize the list so that you could include all the possibilities?
   e) Is this method related to your results in question 1? Explain.
   f) Can you use your method to predict the number of different orders in which eight students could give speeches?

Many counting and probability calculations involve the product of a series of consecutive integers. You can use **factorial** notation to write such expressions more easily. For any natural number n,

$$n! = n \times (n-1) \times (n-2) \times (n-3) \times \ldots \times 3 \times 2 \times 1$$

This expression is read as *n factorial*.

Example 1 Evaluating Factorials

Calculate each factorial.

a) $2!$ **b)** $4!$ **c)** $8!$

Solution

a) $$\begin{aligned} 2! &= 2 \times 1 \\ &= 2 \end{aligned}$$

b) $$\begin{aligned} 4! &= 4 \times 3 \times 2 \times 1 \\ &= 24 \end{aligned}$$

c) $$\begin{aligned} 8! &= 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 \\ &= 40\ 320 \end{aligned}$$

As you can see from Example 1, $n!$ increases dramatically as n becomes larger. However, calculators and computer software provide an easy means of calculating the larger factorials. Most scientific and graphing calculators have a factorial key or function.

Example 2 Using Technology to Evaluate Factorials

Calculate.

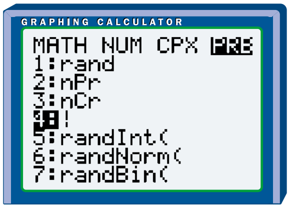
a) 21! **b)** 53! **c)** 70!

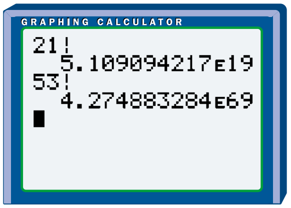
Solution 1 Using a Graphing Calculator

Enter the number on the home screen and then use the **! function** on the MATH PRB menu to calculate the factorial.

a) $21! = 21 \times 20 \times 19 \times 18 \times \ldots \times 2 \times 1$
$= 5.1091 \times 10^{19}$

b) $53! = 53 \times 52 \times 51 \times \ldots \times 3 \times 2 \times 1$
$= 4.2749 \times 10^{69}$

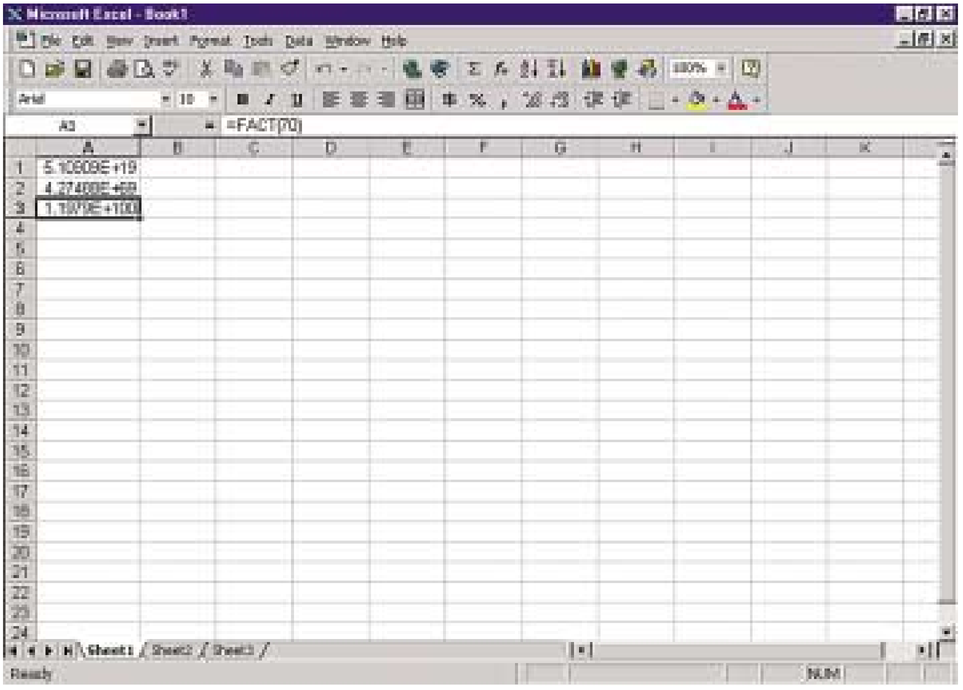




c) Entering 70! on a graphing calculator gives an ERR:OVERFLOW message since $70! > 10^{100}$ which is the largest number the calculator can handle. In fact, 69! is the largest factorial you can calculate directly on TI-83 series calculators.

Solution 2 Using a Spreadsheet

Both Corel® Quattro® Pro and Microsoft® Excel have a built-in **factorial function** with the syntax FACT(*n*).



Example 3 Evaluating Factorial Expressions

Evaluate.

a) $\dfrac{10!}{5!}$ **b)** $\dfrac{83!}{79!}$

Solution

In both these expressions, you can divide out the common terms in the numerator and denominator.

a)
$$\begin{aligned}\frac{10!}{5!} &= \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{5 \times 4 \times 3 \times 2 \times 1}\\ &= 10 \times 9 \times 8 \times 7 \times 6\\ &= 30\ 240\end{aligned}$$

b)
$$\begin{aligned}\frac{83!}{79!} &= \frac{83 \times 82 \times 81 \times 80 \times 79 \times 78 \times \ldots \times 2 \times 1}{79 \times 78 \times \ldots \times 2 \times 1}\\ &= 83 \times 82 \times 81 \times 80\\ &= 44\ 102\ 880\end{aligned}$$

Note that by dividing out the common terms, you can use a calculator to evaluate this expression even though the factorials are too large for the calculator.

Example 4 Counting Possibilities

The senior choir has rehearsed five songs for an upcoming assembly. In how many different orders can the choir perform the songs?

Solution

There are five ways to choose the first song, four ways to choose the second, three ways to choose the third, two ways to choose the fourth, and only one way to choose the final song. Using the fundamental counting principle, the total number of different ways is

$$\begin{aligned}5 \times 4 \times 3 \times 2 \times 1 &= 5!\\ &= 120\end{aligned}$$

The choir can sing the five songs in 120 different orders.

Example 5 Indirect Method

In how many ways could ten questions on a test be arranged, if the easiest question and the most difficult question

a) are side-by-side?

b) are not side-by-side?

Solution

a) Treat the easiest question and the most difficult question as a unit making nine items that are to be arranged. The two questions can be arranged in 2! ways within their unit.

$9! \times 2! = 725\ 760$

The questions can be arranged in 725 760 ways if the easiest question and the most difficult question are side-by-side.

b) Use the indirect method. The number of arrangements with the easiest and most difficult questions separated is equal to the total number of possible arrangements less the number with the two questions side-by-side:

$$\begin{aligned} 10! - 9! \times 2! &= 3\ 628\ 800 - 725\ 760 \\ &= 2\ 903\ 040 \end{aligned}$$

The questions can be arranged in 2 903 040 ways if the easiest question and the most difficult question are not side-by-side.

A **permutation** of n distinct items is an arrangement of all the items in a definite order. The total number of such permutations is denoted by ${}_nP_n$ or $P(n, n)$.

There are n possible ways of choosing the first item, $n - 1$ ways of choosing the second, $n - 2$ ways of choosing the third, and so on. Applying the fundamental counting principle as in Example 5 gives

$$\begin{aligned} {}_nP_n &= n \times (n-1) \times (n-2) \times (n-3) \times \ldots \times 3 \times 2 \times 1 \\ &= n! \end{aligned}$$

Example 6 Applying the Permutation Formula

In how many different orders can eight nominees for the students' council give their speeches at an assembly?

Solution

$$\begin{aligned} {}_8P_8 &= 8! \\ &= 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 \\ &= 40\ 320 \end{aligned}$$

There are 40 320 different orders in which the eight nominees can give their speeches.

Example 7 Student Government

In how many ways could a president and a vice-president be chosen from a group of eight nominees?

Solution

Using the fundamental counting principle, there are 8×7, or 56, ways to choose a president and a vice-president.

A permutation of n distinct items taken r at a time is an arrangement of r of the n items in a definite order. Such permutations are sometimes called r-arrangements of n items. The total number of possible arrangements of r items out of a set of n is denoted by ${}_nP_r$ or $P(n, r)$.

There are n ways of choosing the first item, $n - 1$ ways of choosing the second item, and so on down to $n - r + 1$ ways of choosing the rth item. Using the fundamental counting principle,

$$ {}_nP_r = n(n-1)(n-2)\dots(n-r+1) $$

It is often more convenient to rewrite this expression in terms of factorials.

$$ {}_nP_r = \frac{n!}{(n-r)!} $$

The denominator divides out completely, as in Example 3, so these two ways of writing ${}_nP_r$ are equivalent.

Project Prep

The permutations formula could be a useful tool for your probability project.

Example 8 Applying the Permutation Formula

In a card game, each player is dealt a face down "reserve" of 13 cards that can be turned up and used one by one during the game. How many different sequences of reserve cards could a player have?

Solution 1 Using Pencil and Paper

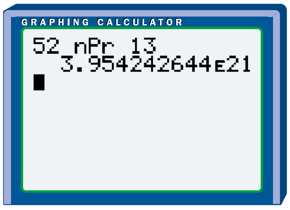
Here, you are taking 13 cards from a deck of 52.

$$ \begin{aligned} {}_{52}P_{13} &= \frac{52!}{(52-13)!} \\ &= \frac{52!}{39!} \\ &= 52 \times 51 \times 50 \times \dots \times 41 \times 40 \\ &= 3.9542 \times 10^{21} \end{aligned} $$

There are approximately 3.95×10^{21} different sequences of reserve cards a player could have.

Solution 2 Using a Graphing Calculator

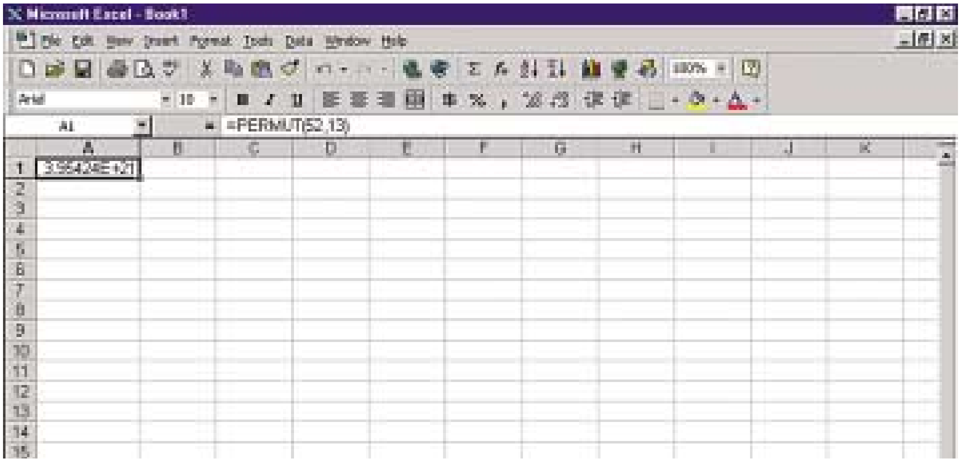
Use the nPr function on the MATH PRB menu.



There are approximately 3.95×10^{21} different sequences of reserve cards a player could turn up during one game.

Solution 3 Using a Spreadsheet

Both Corel® Quattro® Pro and Microsoft® Excel have a **permutations function** with the syntax PERMUT(n,r).



There are approximately 3.95×10^{21} different sequences of reserve cards a player could turn up during one game.

Key Concepts

- A factorial indicates the multiplication of consecutive natural numbers. $n! = n(n-1)(n-2) \times \ldots \times 1$.
- The number of permutations of n distinct items chosen n at a time in a definite order is ${}_nP_n = n!$
- The number of permutations of r items taken from n distinct items is ${}_nP_r = \dfrac{n!}{(n-r)!}$.

Communicate Your Understanding

1. Explain why it is convenient to write the expression for the number of possible permutations in terms of factorials.

2. **a)** Is $(-3)!$ possible? Explain your answer.

b) In how many ways can you order an empty list, or zero items? What does this tell you about the value of 0!? Check your answer using a calculator.

Practise

A

1. Express in factorial notation.

a) $6 \times 5 \times 4 \times 3 \times 2 \times 1$

b) $8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$

c) $3 \times 2 \times 1$

d) $9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$

2. Evaluate.

a) $\frac{7!}{4!}$ **b)** $\frac{11!}{9!}$

c) $\frac{8!}{5!\,2!}$ **d)** $\frac{15!}{3!\,8!}$

e) $\frac{85!}{82!}$ **f)** $\frac{14!}{4!\,5!}$

3. Express in the form ${}_nP_r$.

a) $6 \times 5 \times 4$

b) $9 \times 8 \times 7 \times 6$

c) $20 \times 19 \times 18 \times 17$

d) $101 \times 100 \times 99 \times 98 \times 97$

e) $76 \times 75 \times 74 \times 73 \times 72 \times 71 \times 70$

4. Evaluate without using technology.

a) $P(10, 4)$ **b)** $P(16, 4)$ **c)** ${}_5P_2$

d) ${}_9P_4$ **e)** $7!$

5. Use either a spreadsheet or a graphing or scientific calculator to verify your answers to question 4.

Apply, Solve, Communicate

6. **a)** How many ways can you arrange the letters in the word *factor*?

b) How many ways can Ismail arrange four different textbooks on the shelf in his locker?

c) How many ways can Laura colour 4 adjacent regions on a map if she has a set of 12 coloured pencils?

B

7. Simplify each of the following in factorial form. Do not evaluate.

a) $12 \times 11 \times 10 \times 9!$

b) $72 \times 7!$

c) $(n + 4)(n + 5)(n + 3)!$

8. **Communication** Explain how a factorial is an iterative process.

9. Seven children are to line up for a photograph.

a) How many different arrangements are possible?

b) How many arrangements are possible if Brenda is in the middle?

c) How many arrangements are possible if Ahmed is on the far left and Yen is on the far right?

d) How many arrangements are possible if Hanh and Brian must be together?

10. A 12-volume encyclopedia is to be placed on a shelf. How many incorrect arrangements are there?

11. In how many ways can the 12 members of a volleyball team line up, if the captain and assistant captain must remain together?

12. Ten people are to be seated at a rectangular table for dinner. Tanya will sit at the head of the table. Henry must not sit beside either Wilson or Nancy. In how many ways can the people be seated for dinner?

13. **Application** Joanne prefers classical and pop music. If her friend Charlene has five classical CDs, four country and western CDs, and seven pop CDs, in how many orders can Joanne and Charlene play the CDs Joanne likes?

14. In how many ways can the valedictorian, class poet, and presenter of the class gift be chosen from a class of 20 students?

15. **Application** If you have a standard deck of 52 cards, in how many different ways can you deal out

a) 5 cards? **b)** 10 cards?

c) 5 red cards? **d)** 4 queens?

16. **Inquiry / Problem Solving** Suppose you are designing a coding system for data relayed by a satellite. To make transmissions errors easier to detect, each code must have no repeated digits.

a) If you need 60 000 different codes, how many digits long should each code be?

b) How many ten-digit codes can you create if the first three digits must be 1, 3, or 6?

17. Arnold Schoenberg (1874–1951) pioneered serialism, a technique for composing music based on a tone row, a sequence in which each of the 12 tones in an octave is played only once. How many tone rows are possible?

18. Consider the students' council described on page 223 at the beginning of this chapter.

Chapter Problem

a) In how many ways can the secretary, treasurer, social convenor, and fundraising chair be elected if all ten nominees are eligible for any of these positions?

b) In how many ways can the council be chosen if the president and vice-president must be grade 12 students and the grade representatives must represent their current grade level?

19. **Inquiry / Problem Solving** A student has volunteered to photograph the school's championship basketball team for the yearbook. In order to get the perfect picture, the student plans to photograph the ten players and their coach lined up in every possible order. Determine whether this plan is practical.

ACHIEVEMENT CHECK

Knowledge / Understanding	Thinking / Inquiry / Problem Solving	Communication	Application

20. Wayne has a briefcase with a three-digit combination lock. He can set the combination himself, and his favourite digits are 3, 4, 5, 6, and 7. Each digit can be used at most once.

a) How many permutations of three of these five digits are there?

b) If you think of each permutation as a three-digit number, how many of these numbers would be odd numbers?

c) How many of the three-digit numbers are even numbers and begin with a 4?

d) How many of the three-digit numbers are even numbers and do *not* begin with a 4?

e) Is there a connection among the four answers above? If so, state what it is and why it occurs.

C

21. TI-83 series calculators use the definition $\left(-\frac{1}{2}\right)! = \sqrt{\pi}$. Research the origin of this definition and explain why it is useful for mathematical calculations.

22. **Communication** How many different ways can six people be seated at a round table? Explain your reasoning.

23. What is the highest power of 2 that divides evenly into 100! ?

24. A committee of three teachers are to select the winner from among ten students nominated for special award. The teachers each make a list of their top three choices in order. The lists have only one name in common, and that name has a different rank on each list. In how many ways could the teachers have made their lists?