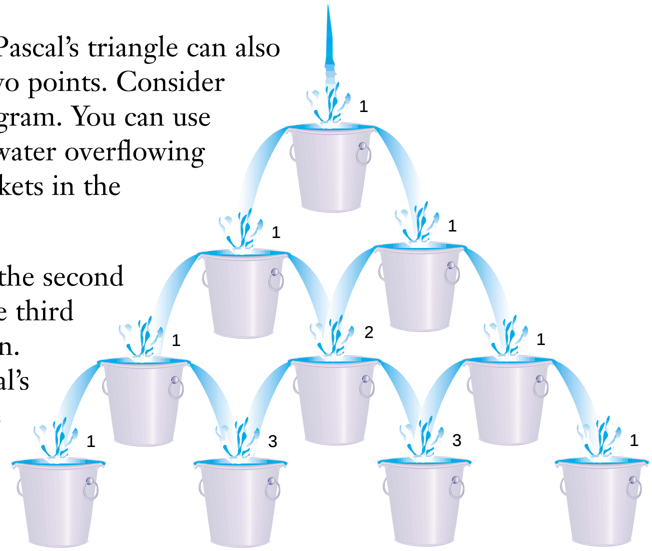
4.5 Applying Pascal's Method

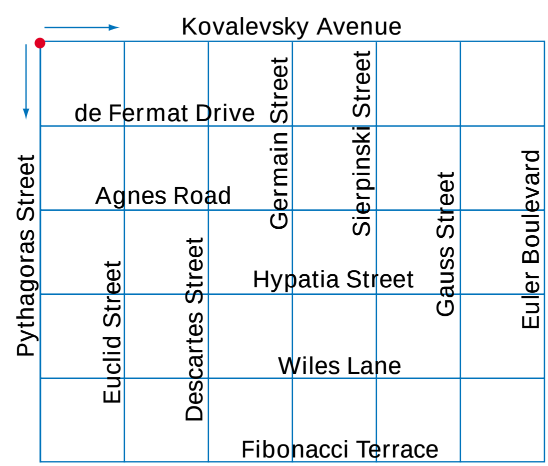
The iterative process that generates the terms in Pascal's triangle can also be applied to counting paths or routes between two points. Consider water being poured into the top bucket in the diagram. You can use Pascal's method to count the different paths that water overflowing from the top bucket could take to each of the buckets in the bottom row.



The water has one path to each of the buckets in the second row. There is one path to each outer bucket of the third row, but two paths to the middle bucket, and so on. The numbers in the diagram match those in Pascal's triangle because they were derived using the same method—Pascal's method.

INVESTIGATE & INQUIRE: Counting Routes

Suppose you are standing at the corner of Pythagoras Street and Kovalevsky Avenue, and want to reach the corner of Fibonacci Terrace and Euler Boulevard. To avoid going out of your way, you would travel only east and south. Notice that you could start out by going to the corner of either Euclid Street and Kovalevsky Avenue or Pythagoras Street and de Fermat Drive.



1. How many routes are possible to the corner of Euclid Street and de Fermat Drive from your starting point? Sketch the street grid and mark the number of routes onto it.
2. a) Continue to travel only east or south. How many routes are possible from the start to the corner of
      i) Descartes Street and Kovalevsky Avenue?
      ii) Pythagoras Street and Agnes Road?
      iii) Euclid Street and Agnes Road?
      iv) Descartes Street and de Fermat Drive?
      v) Descartes Street and Agnes Road?
   b) List the routes you counted in part a).

3. Consider your method and the resulting numbers. How do they relate to Pascal's triangle?
4. Continue to mark the number of routes possible on your sketch until you have reached the corner of Fibonacci Terrace and Euler Boulevard. How many different routes are possible?
5. Describe the process you used to find the number of routes from Pythagoras Street and Kovalevsky Avenue to Fibonacci Terrace and Euler Boulevard.

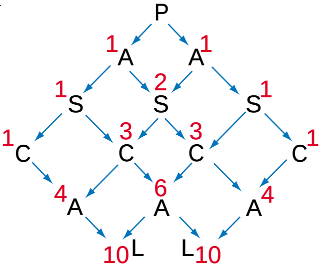
Example 1 Counting Paths in an Array

Determine how many different paths will spell *PASCAL* if you start at the top and proceed to the next row by moving diagonally left or right.

```
      P
    A   A
  S   S   S
C   C   C   C
  A   A   A
    L   L
```

Solution

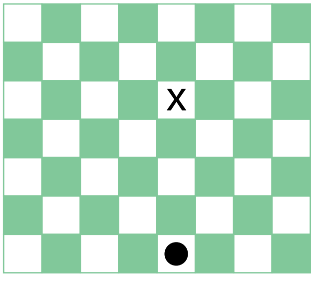
Starting at the top, record the number of possible paths moving diagonally to the left and right as you proceed to each different letter. For instance, there is one path from *P* to the left *A* and one path from *P* to the right *A*. There is one path from an *A* to the left *S*, two paths from an *A* to the middle *S*, and one path from an *A* to the right *S*.



Continuing with this counting reveals that there are 10 different paths leading to each *L*. Therefore, a total of 20 paths spell *PASCAL*.

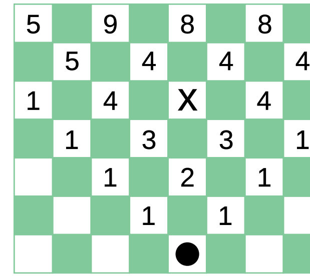
Example 2 Counting Paths on a Checkerboard

On the checkerboard shown, the checker can travel only diagonally upward. It cannot move through a square containing an X. Determine the number of paths from the checker's current position to the top of the board.



Solution

Use Pascal's method to find the number of paths to each successive position. There is one path possible into each of the squares diagonally adjacent to the checker's starting position. From the second row there are four paths to the third row: one path to the third square from the left, two to the fifth square, and one to the seventh square. Continue this process for the remaining four rows. The square containing an X gets a zero or no number since there are no paths through this blocked square.



From left to right, there are 5, 9, 8, and 8 paths to the white squares at the top of the board, making a total of 30 paths.

Key Concepts

- Pascal's method involves adding two neighbouring terms in order to find the term below.
- Pascal's method can be applied to counting paths in a variety of arrays and grids.

Communicate Your Understanding

1. Suggest a context in which you could apply Pascal's method, other than those in the examples above.
2. Which of the numbers along the perimeter of a map tallying possible routes are always 1? Explain.

Practise

A

1. Fill in the missing numbers using Pascal's method.

				495		
					825	
		3003		2112		

2. In the following arrangements of letters, start from the top and proceed to the next row by moving diagonally left or right. How many different paths will spell each word?

a)

P
A A
T T T
T T T T
E E E E E
R R R R R R
N N N N N N N
S S S S S S S S

b)

M
A A
T T T
H H H H
E E E E E
M M M M M M
A A A A A A A
T T T T T T
I I I I I
C C C C
S S S

c)

T
R R
I I I
A A A A
N N N
G G
L L L
E E E E

3. The first nine terms of a row of Pascal's triangle are shown below. Determine the first nine terms of the previous and next rows.

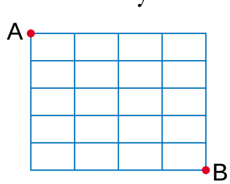
1 16 120 560 1820 4368 8008 11 440 12 870

Apply, Solve, Communicate

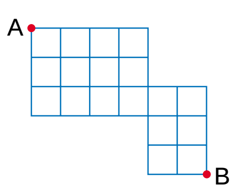
B

4. Determine the number of possible routes from A to B if you travel only south or east.

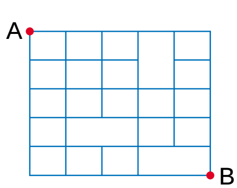
a)



b)



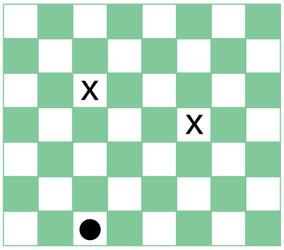
c)



5. Sung is three blocks east and five blocks south of her friend's home. How many different routes are possible if she walks only west or north?

6. Ryan lives four blocks north and five blocks west of his school. Is it possible for him to take a different route to school each day, walking only south and east? Assume that there are 194 school days in a year.

7. A checker is placed on a checkerboard as shown. The checker may move diagonally upward. Although it cannot move into a square with an X, the checker may jump over the X into the diagonally opposite square.

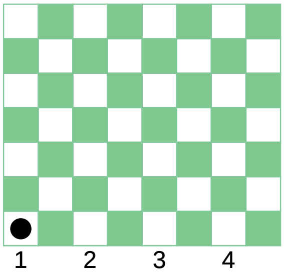


a) How many paths are there to the top of the board?

b) How many paths would there be if the checker could move both diagonally and straight upward?

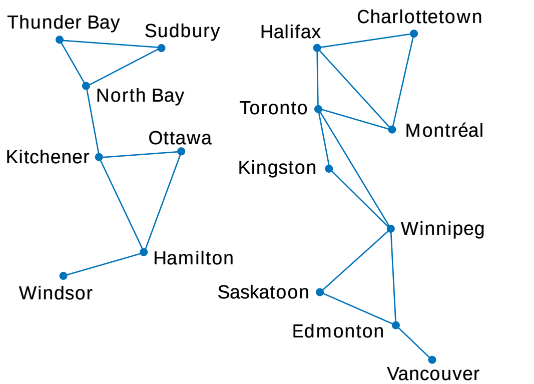
8. **Inquiry/Problem Solving**

a) If a checker is placed as shown below, how many possible paths are there for that checker to reach the top of the game board? Recall that checkers can travel only diagonally on the white squares, one square at a time, moving upward.



b) When a checker reaches the opposite side, it becomes a "king." If the starting squares are labelled 1 to 4, from left to right, from which starting square does a checker have the most routes to become a king? Verify your statement.

9. Application The following diagrams represent communication networks between a company's computer centres in various cities.



a) How many routes are there from Windsor to Thunder Bay?

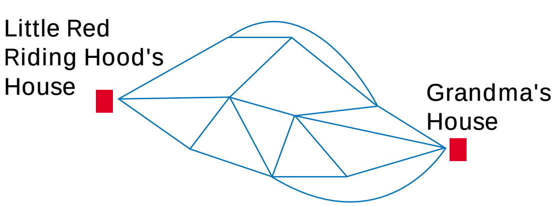
b) How many routes are there from Ottawa to Sudbury?

c) How many routes are there from Montréal to Saskatoon?

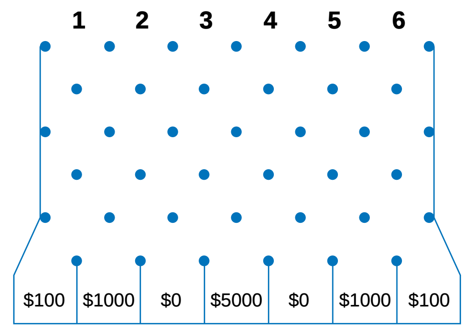
d) How many routes are there from Vancouver to Charlottetown?

e) If the direction were reversed, would the number of routes be the same for parts a) to d)? Explain.

10. To outfox the Big Bad Wolf, Little Red Riding Hood mapped all the paths through the woods to Grandma's house. How many different routes could she take, assuming she always travels from left to right?



11. Communication A popular game show uses a more elaborate version of the Plinko board shown below. Contestants drop a peg into one of the slots at the top of the upright board. The peg is equally likely to go left or right at each post it encounters.



a) Into which slot should contestants drop their pegs to maximize their chances of winning the $5000 prize? Which slot gives contestants the least chance of winning this prize? Justify your answers.

b) Suppose you dropped 100 pegs into the slots randomly, one at a time. Sketch a graph of the number of pegs likely to wind up in each compartment at the bottom of the board. How is this graph related to those described in earlier chapters?

12. Inquiry/Problem Solving

a) Build a new version of Pascal's triangle, using the formula for $t_{n,r}$ on page 247, but start with $t_{0,0} = 2$.

b) Investigate this triangle and state a conjecture about its terms.

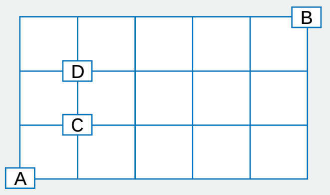
c) State a conjecture about the sum of the terms in each row.

13. Inquiry/Problem Solving Develop a formula relating $t_{n,r}$ of Pascal's triangle to the terms in row $n - 3$.

ACHIEVEMENT CHECK

Knowledge / Understanding	Thinking / Inquiry / Problem Solving	Communication	Application

14. The grid below shows the streets in Anya's neighbourhood.



a) If she only travels east and north, how many different routes can Anya take from her house at intersection A to her friend's house at intersection B?

b) How many of the routes in part a) have only one change of direction?

c) Suppose another friend lives at intersection C. How many ways can Anya travel from A to B, meeting her friend at C along the way?

d) How many ways can she travel to B without passing through C? Explain your reasoning.

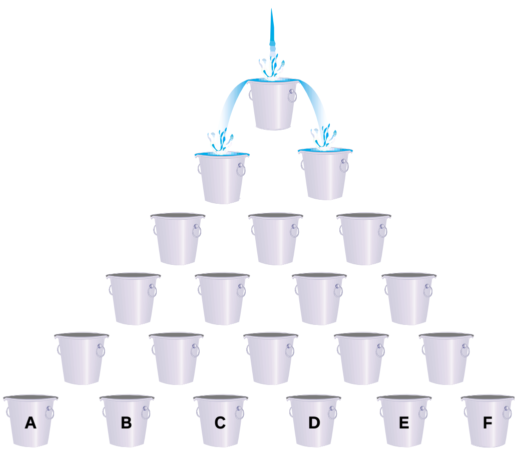
e) If Anya takes any route from A to B, is she more likely to pass through intersection C or D? Explain your reasoning.

C

15. Develop a general formula to determine the number of possible routes to travel n blocks north and m blocks west.

16. **Inquiry / Problem Solving** In chess, a knight moves in L-shaped jumps consisting of two squares along a row or column plus one square at a right angle. On a standard 8×8 chessboard, the starting position for a knight is the second square of the bottom row. If the knight travels upward on every move, how many routes can it take to the top of the board?

17. **Inquiry / Problem Solving** Water is poured into the top bucket of a triangular stack of 2-L buckets. When each bucket is full, the water overflows equally on both sides into the buckets immediately below. How much water will have been poured into the top bucket when at least one of the buckets in the bottom row is full?



18. **Application** Is it possible to arrange a pyramid of buckets such that the bottom layer will fill evenly when water overflows from the bucket at the top of the pyramid?

19. **Application** Enya is standing in the centre square of a 9 by 9 grid. She travels outward one square at a time, moving diagonally or along a row or column. How many different paths can Enya follow to the perimeter?

20. **Communication** Describe how a chessboard path activity involving Pascal's method is related to network diagrams like those in section 1.5. Would network diagrams for such activities be planar? Explain.