5.2 Combinations

In Chapter 4, you learned about permutations—arrangements in which the order of the items is specified. However, in many situations, order does not matter. For example, in many card games, what is in your hand is important, but the order in which it was dealt is not.

INVESTIGATE & INQUIRE: Students' Council

Suppose the students at a secondary school elect a council of eight members, two from each grade. This council then chooses two of its members as co-chairpersons. How could you calculate the number of different pairs of members who could be chosen as the co-chairs?

Choose someone in the class to record your answers to the following questions on a blackboard or an overhead projector.

a) Start with the simplest case. Choose two students to stand at the front of the class. In how many ways can you choose two co-chairs from this pair of students?

b) Choose three students to be at the front of the class. In how many ways can you choose two co-chairs from this trio?

c) In how many ways can you choose two co-chairs from a group of four students?

d) In how many ways can you choose two co-chairs from a group of five students? Do you see a pattern developing? If so, what is it? If not, try choosing from a group of six students and then from a group of seven students while continuing to look for a pattern.

e) When you see a pattern, predict the number of ways two co-chairs can be chosen from a group of eight students.

f) Can you suggest how you could find the answers for this investigation from the numbers of permutations you found in the investigation in section 4.2?

In the investigation on the previous page, you were dealing with a situation in which you were selecting two people from a group, but the order in which you chose the two did not matter. In a permutation, there is a difference between selecting, say, Bob as president and Margot as vice-president as opposed to selecting Margot as president and Bob as vice-president. If you select Bob and Margot as co-chairs, the order in which you select them does not matter since they will both have the same job.

A selection from a group of items without regard to order is called a **combination**.

Example 1 Comparing Permutations and Combinations

a) In how many ways could Alana, Barbara, Carl, Domenic, and Edward fill the positions of president, vice-president, and secretary?

b) In how many ways could these same five people form a committee with three members? List the ways.

c) How are the numbers of ways in parts a) and b) related?

Solution

a) Since the positions are different, order is important. Use a permutation, ${}_nP_r$. There are five people to choose from, so $n = 5$. There are three people being chosen, so $r = 3$. The number of permutations is ${}_5P_3 = 60$.

There are 60 ways Alana, Barbara, Carl, Domenic, and Edward could fill the positions of president, vice-president, and secretary.

b) The easiest way to find all committee combinations is to write them in an ordered fashion. Let A represent Alana, B represent Barbara, C represent Carl, D represent Domenic, and E represent Edward.

The possible combinations are:

A B C	A B D	A B E	A C D	A C E
A D E	B C D	B C E	B D E	C D E

All other possible arrangements include the same three people as one of the combinations listed above. For example, ABC is the same as ACB, BAC, BCA, CAB, and CBA since order is not important.

So, there are only ten ways Alana, Barbara, Carl, Domenic and Edward can form a three-person committee.

c) In part a), there were 60 possible permutations, while in part b), there were 10 possible combinations. The difference is a factor of 6. This factor is ${}_3P_3 = 3!$, the number of possible arrangements of the three people in each combination. Thus,

$$\begin{aligned}\text{number of combinations} &= \frac{\text{number of permutations}}{\text{number of permutations of the objects selected}}\\ &= \frac{{}_5P_3}{3!}\\ &= \frac{60}{6}\\ &= 10\end{aligned}$$

Combinations of *n* distinct objects taken *r* at a time

The number of combinations of r objects chosen from a set of n distinct objects is

$$\begin{aligned}{}_nC_r &= \frac{{}_nP_r}{r!}\\ &= \frac{\dfrac{n!}{(n-r)!}}{r!}\\ &= \frac{n!}{(n-r)!r!}\end{aligned}$$

The notations ${}_nC_r$, $C(n, r)$, and $\binom{n}{r}$ are all equivalent. Many people prefer the form $\binom{n}{r}$ when a number of combinations are multiplied together. The symbol ${}_nC_r$ is used most often in this text since it is what appears on most scientific and graphing calculators.

Example 2 Applying the Combinations Formula

How many different sampler dishes with 3 different flavours could you get at an ice-cream shop with 31 different flavours?

Solution

There are 31 flavours, so $n = 31$. The sampler dish has 3 flavours, so $r = 3$.

$$\begin{aligned}{}_{31}C_3 &= \frac{31!}{(31-3)!\,3}\\ &= \frac{31!}{28!3!}\\ &= \frac{31 \times 30 \times 29}{3 \times 2}\\ &= 4495\end{aligned}$$

There are 4495 possible sampler combinations.

Note that the number of combinations in Example 2 was easy to calculate because the number of items chosen, r, was quite small.

Example 3 Calculating Numbers of Combinations Manually

A ballet choreographer wants 18 dancers for a scene.

a) In how many ways can the choreographer choose the dancers if the company has 20 dancers? 24 dancers?

b) How would you advise the choreographer to choose the dancers?

Solution

a) When n and r are close in value, ${}_nC_r$ can be calculated mentally.
With $n = 20$ and $r = 18$,

$$
\begin{aligned}
{}_{20}C_{18} &= \frac{20!}{(20-18)!\,18!} \\
&= \frac{20 \times 19}{2!} && 20 \div 2 = 10 \\
&= 190 && \text{Then, } 10 \times 19 = 190
\end{aligned}
$$

The choreographer could choose from 190 different combinations of the 20 dancers.

With $n = 24$ and $r = 18$, ${}_nC_r$ can be calculated manually or with a basic calculator once you have divided out the common terms in the factorials.

$$
\begin{aligned}
{}_{24}C_{18} &= \frac{24!}{(24-18)!\,18!} \\
&= \frac{24 \times 23 \times 22 \times 21 \times 20 \times 19}{6!} \\
&= \frac{24 \times 23 \times 22 \times 21 \times 20 \times 19}{6 \times 5 \times 4 \times 3 \times 2 \times 1} \\
&= 23 \times 11 \times 7 \times 4 \times 19 \\
&= 134\,596
\end{aligned}
$$

With the 4 additional dancers, the choreographer now has a choice of 134 596 combinations.

b) From part a), you can see that it would be impractical for the choreographer to try every possible combination. Instead the choreographer could use an indirect method and try to decide which dancers are least likely to be suitable for the scene.

Even though there are fewer permutations of n objects than there are combinations, the numbers of combinations are often still too large to calculate manually.

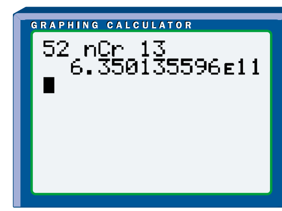
Example 4 Using Technology to Calculate Numbers of Combinations

Each player in a bridge game receives a hand of 13 cards dealt from a standard deck. How many different bridge hands are possible?

For details of calculator and software functions, refer to Appendix B.

Solution 1 Using a Graphing Calculator

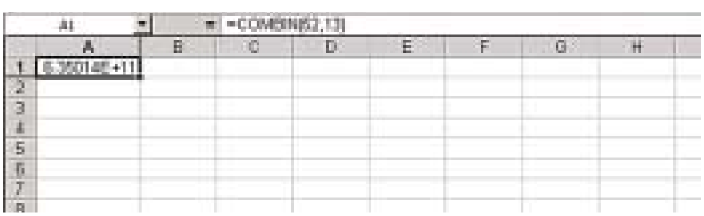
Here, the order in which the player receives the cards does not matter. What you want to determine is the number of different combinations of cards a player could have once the dealing is complete. So, the answer is simply ${}_{52}C_{13}$. You can evaluate ${}_{52}C_{13}$ by using the **nCr function** on the MATH PRB menu of a graphing calculator. This function is similar to the **nPr function** used for permutations.



There are about 635 billion possible bridge hands.

Solution 2 Using a Spreadsheet

Most spreadsheet programs have a **combinations function** for calculating numbers of combinations. In Microsoft® Excel, this function is the COMBIN(n,r) function. In Corel® Quattro® Pro, this function is the COMB(r,n) function.



You now have a variety of methods for finding numbers of combinations—paper-and-pencil calculations, factorials, scientific or graphing calculators, and software. When appropriate, you can also apply both of the counting principles described in Chapter 4.

Project Prep

Techniques for calculating numbers of combinations could be helpful for designing the game in your probability project, especially if your game uses cards.

Example 5 Using the Counting Principles With Combinations

Ursula runs a small landscaping business. She has on hand 12 kinds of rose bushes, 16 kinds of small shrubs, 11 kinds of evergreen seedlings, and 18 kinds of flowering lilies. In how many ways can Ursula fill an order if a customer wants

a) 15 different varieties consisting of 4 roses, 3 shrubs, 2 evergreens, and 6 lilies?

b) either 4 different roses or 6 different lilies?

Solution

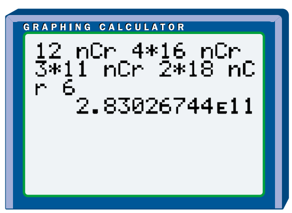
a) The order in which Ursula chooses the plants does not matter.

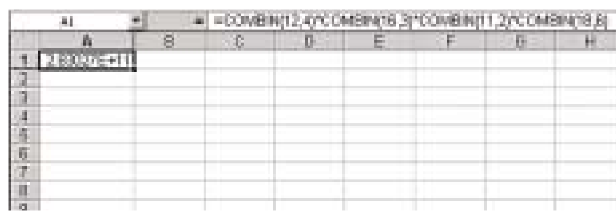
The number of ways of choosing the roses is ${}_{12}C_4$.
The number of ways of choosing the shrubs is ${}_{16}C_3$.
The number of ways of choosing the evergreens is ${}_{11}C_2$.
The number of ways of choosing the lilies is ${}_{18}C_6$.

Since varying the rose selection for each different selection of the shrubs, evergreens, and lilies produces a different choice of plants, you can apply the fundamental (multiplicative) counting principle. Multiply the series of combinations to find the total number of possibilities.

$$\begin{aligned} {}_{12}C_4 \times {}_{16}C_3 \times {}_{11}C_2 \times {}_{18}C_6 &= 495 \times 560 \times 55 \times 18\,564 \\ &= 2.830\,267\,44 \times 10^{11} \end{aligned}$$

Ursula has over 283 billion ways of choosing the plants for her customer.





b) Ursula can choose the 4 rose bushes in ${}_{12}C_4$ ways.

She can choose the 6 lilies in ${}_{18}C_6$ ways.

Since the customer wants *either* the rose bushes *or* the lilies, you can apply the additive counting principle to find the total number of possibilities.

$$\begin{aligned} {}_{12}C_4 + {}_{18}C_6 &= 495 + 18\,564 \\ &= 19\,059 \end{aligned}$$

Ursula can fill the order for either roses or lilies in 19 059 ways.

As you can see, even relatively simple situations can produce very large numbers of combinations.

Key Concepts

- A combination is a selection of objects in which order is not important.
- The number of combinations of n distinct objects taken r at a time is denoted as ${}_nC_r$, $C(n, r)$, or $\binom{n}{r}$ and is equal to $\dfrac{n!}{(n-r)!\,r!}$.
- The multiplicative and additive counting principles can be applied to problems involving combinations.

Communicate Your Understanding

1. Explain why n objects have more possible permutations than combinations. Use a simple example to illustrate your explanation.
2. Explain whether you would use combinations, permutations, or another method to calculate the number of ways of choosing
   a) three items from a menu of ten items
   b) an appetizer, an entrée, and a dessert from a menu with three appetizers, four entrées, and five desserts
3. Give an example of a combination expression you could calculate
   a) by hand
   b) algebraically
   c) only with a calculator or computer

Practise

A

1. Evaluate using a variety of methods. Explain why you chose each method.
   a) ${}_{21}C_{19}$ b) ${}_{30}C_{28}$
   c) ${}_{18}C_{5}$ d) ${}_{16}C_{3}$
   e) ${}_{19}C_{4}$ f) ${}_{25}C_{20}$
2. Evaluate the following pairs of combinations and compare their values.
   a) ${}_{11}C_{1}$, ${}_{11}C_{10}$
   b) ${}_{11}C_{2}$, ${}_{11}C_{9}$
   c) ${}_{11}C_{3}$, ${}_{11}C_{8}$

Apply, Solve, Communicate

B

3. **Communication** In how many ways could you choose 2 red jellybeans from a package of 15 red jellybeans? Explain your reasoning.
4. How many ways can 4 cards be chosen from a deck of 52, if the order in which they are chosen does not matter?
5. How many groups of 3 toys can a child choose to take on a vacation from a toy box containing 11 toys?
6. How many sets of 6 questions for a test can be chosen from a list of 22 questions?
7. In how many ways can a teacher select 5 students from a class of 23 to make a bulletin-board display? Explain your reasoning.
8. As a promotion, a video store decides to give away posters for recently released movies.
   a) If posters are available for 27 recent releases, in how many ways could the video-store owner choose 8 different posters for the promotion?
   b) Are you able to calculate the number of ways mentally? Why or why not?

9. **Communication** A club has 11 members.
   - a) How many different 2-member committees could the club form?
   - b) How many different 3-member committees could the club form?
   - c) In how many ways can a club president, treasurer, and secretary be chosen?
   - d) By what factor do the answers in parts b) and c) differ? How do you account for this difference?

10. Fritz has a deck of 52 cards, and he may choose any number of these cards, from none to all. Use a spreadsheet or Fathom™ to calculate and graph the number of combinations for each of Fritz's choices.

11. **Application** A track club, a swim club, and a cycling club are forming a joint committee to organize a triathlon. The committee will have two members from each club. In how many ways can the committee be formed if ten runners, eight swimmers, and seven cyclists volunteer to serve on it?

12. In how many ways can a jury of 6 men and 6 women be chosen from a group of 10 men and 15 women?

13. **Inquiry / Problem Solving** There are 15 technicians and 11 chemists working in a research laboratory. In how many ways could they form a 5-member safety committee if the committee
   - a) may be chosen in any way?
   - b) must have exactly one technician?
   - c) must have exactly one chemist?
   - d) must have exactly two chemists?
   - e) may be all technicians or all chemists?

14. Jeffrey, a DJ at a local radio station, is choosing the music he will play on his shift. He must choose all his music from the top 100 songs for the week and he must play at least 12 songs an hour. In his first hour, all his choices must be from the top-20 list.
   - a) In how many ways can Jeffrey choose the songs for his first hour if he wants to choose exactly 12 songs?
   - b) In how many ways can Jeffrey choose the 12 songs if he wants to pick 8 of the top 10 and 4 from the songs listed from 11 to 20 on the chart?
   - c) In how many ways can Jeffrey choose either 12 or 13 songs to play in the first hour of his shift?
   - d) In how many ways can Jeffrey choose the songs if he wants to play up to 15 songs in the first hour?

15. The game of euchre uses only 24 of the cards from a standard deck. How many different five-card euchre hands are possible?

16. **Application** A taxi is shuttling 11 students to a concert. The taxi can hold only 4 students. In how many ways can 4 students be chosen for
   - a) the taxi's first trip?
   - b) the taxi's second trip?

17. Diane is making a quilt. She needs three pieces with a yellow undertone, two pieces with a blue undertone, and four pieces with a white undertone. If she has six squares with a yellow undertone, five with a blue undertone, and eight with a white undertone to choose from, in how many ways can she choose the squares for the quilt?

18. **Inquiry/Problem Solving** At a family reunion, everyone greets each other with a handshake. If there are 20 people at the reunion, how many handshakes take place?

ACHIEVEMENT CHECK

Knowledge/Understanding	Thinking/Inquiry/Problem Solving	Communication	Application

19. A basketball team consists of five players—one centre, two forwards, and two guards. The senior squad at Vennville Central High School has two centres, six forwards, and four guards.

a) How many ways can the coach pick the two starting guards for a game?

b) How many different starting lineups are possible if all team members play their specified positions?

c) How many of these starting lineups include Dana, the team's 185-cm centre?

d) Some coaches designate the forwards as power forward and small forward. If all six forwards are adept in either position, how would this designation affect the number of possible starting lineups?

e) As the league final approaches, the centre Dana, forward Ashlee, and guard Hollie are all down with a nasty flu. Fortunately, the five healthy forwards can also play the guard position. If the coach can assign these players as either forwards or guards, will the number of possible starting lineups be close to the number in part b)? Support your answer mathematically.

f) Is the same result achieved if the forwards play their regular positions but the guards can play as either forwards or guards? Explain your answer.

20. In the game of bridge, each player is dealt a hand of 13 cards from a standard deck of 52 cards.

a) By what factor does the number of possible bridge hands differ from the number of ways a bridge hand could be dealt to a player? Explain your reasoning.

b) Use combinations to write an expression for the number of bridge hands that have exactly five clubs, two spades, three diamonds, and three hearts.

c) Use combinations to write an expression for the number of bridge hands that have exactly five hearts.

d) Use software or a calculator to evaluate the expressions in parts b) and c).

C

21. There are 18 students involved in the class production of *Arsenic and Old Lace*.

a) In how many ways can the teacher cast the play if there are five male roles and seven female roles and the class has nine male and nine female students?

b) In how many ways can the teacher cast the play if Jean will play the young female part only if Jovane plays the male lead?

c) In how many ways can the teacher cast the play if all the roles could be played by either a male or a female student?

22. A large sack contains six basketballs and five volleyballs. Find the number of combinations of four balls that can be chosen from the sack if

a) they may be any type of ball

b) two must be volleyballs and two must be basketballs

c) all four must be volleyballs

d) none may be volleyballs