5.3 Problem Solving With Combinations

In the last section, you considered the number of ways of choosing r items from a set of n distinct items. This section will examine situations where you want to know the total number of possible combinations of any size that you could choose from a given number of items, some of which may be identical.

INVESTIGATE & INQUIRE: Combinations of Coins

1. a) How many different sums of money can you create with a penny and a nickel? List these sums.
   b) How many different sums can you create with a penny, a nickel, and a dime? List them.
   c) Predict how many different sums you can create with a penny, a nickel, a dime, and a quarter. Test your conjecture by listing the possible sums.
2. a) How many different sums of money can you create with two pennies and a dime? List them.
   b) How many different sums can you create with three pennies and a dime?
   c) Predict how many sums you can create with four pennies and a dime. Test your conjecture. Can you see a pattern developing? If so, what is it?

Example 1 All Possible Combinations of Distinct Items

An artist has an apple, an orange, and a pear in his refrigerator. In how many ways can the artist choose one or more pieces of fruit for a still-life painting?

Solution

The artist has two choices for each piece of fruit: either include it in the painting or leave it out. Thus, the artist has a total of $2 \times 2 \times 2 = 8$ choices. Note that one of these choices is to leave out the apple, the orange, *and* the pear. However, the artist wants at least one piece of fruit in his painting. Thus, he has $2^3 - 1 = 7$ combinations to choose from.

You can apply the same logic to any group of distinct items.

The total number of combinations containing at least one item chosen from a group of n distinct items is $2^n - 1$.

Remember that combinations are subsets of the group of n objects. A **null set** is a set that has no elements. Thus,

A set with n distinct elements has 2^n subsets including the null set.

Example 2 Applying the Formula for Numbers of Subsets

In how many ways can a committee with at least one member be appointed from a board with six members?

Solution

The board could choose 1, 2, 3, 4, 5, or 6 people for the committee, so $n = 6$. Since the committee must have at least one member, use the formula that excludes the null set.

$$\begin{aligned} 2^6 - 1 &= 64 - 1 \\ &= 63 \end{aligned}$$

There are 63 ways to choose a committee of at least one person from a six-member board.

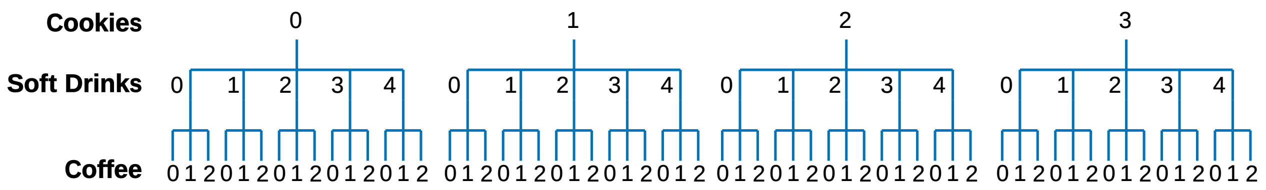
Example 3 All Possible Combinations With Some Identical Items

Kate is responsible for stocking the coffee room at her office. She can purchase up to three cases of cookies, four cases of soft drinks, and two cases of coffee packets without having to send the order through the accounting department. How many different direct purchases can Kate make?

Solution

Kate can order more than one of each kind of item, so this situation involves combinations in which some items are alike.

- Kate may choose to buy three or two or one or no cases of cookies, so she has four ways to choose cookies.
- Kate may choose to buy four or three or two or one or no cases of soft drinks, so she has five ways to choose soft drinks.
- Kate may choose to buy two or one or no cases of coffee packets, so she has three ways to choose coffee.



As shown on the first branch of the diagram above, one of these choices is purchasing *no* cookies, *no* soft drinks, and *no* coffee. Since this choice is not a purchase at all, subtract it from the total number of choices.

Thus, Kate can make $4 \times 5 \times 3 - 1 = 59$ different direct purchases.

In a situation where you can choose all, some, or none of the p items available, you have $p + 1$ choices. You can then apply the fundamental (multiplicative) counting principle if you have successive choices of different kinds of items. Always consider whether the choice of not picking any items makes sense. If it does not, subtract 1 from the total.

Combinations of Items in Which Some are Alike

If at least one item is chosen, the total number of selections that can be made from p items of one kind, q items of another kind, r items of another kind, and so on, is $(p + 1)(q + 1)(r + 1) \ldots - 1$

Having identical elements in a set reduces the number of possible combinations when you choose r items from that set. You cannot calculate this number by simply dividing by a factorial as you did with permutations in section 4.3. Often, you have to consider a large number of cases individually. However, some situations have restrictive conditions that make it much easier to count the number of possible combinations.

Example 4 Combinations With Some Identical Items

The director of a short documentary has found five rock songs, two blues tunes, and three jazz pieces that suit the theme of the film. In how many ways can the director choose three pieces for the soundtrack if she wants the film to include some jazz?

Solution 1 Counting Cases

The director can select exactly one, two, or three jazz pieces.

Case 1: One jazz piece
The director can choose the one jazz piece in ${}_3C_1$ ways and two of the seven non-jazz pieces in ${}_7C_2$ ways. Thus, there are ${}_3C_1 \times {}_7C_2 = 63$ combinations of music with one jazz piece.

Case 2: Two jazz pieces
The director can choose the two jazz pieces in ${}_3C_2$ ways and one of the seven non-jazz pieces in ${}_7C_1$ ways. There are ${}_3C_2 \times {}_7C_1 = 21$ combinations with two jazz pieces.

Case 3: Three jazz pieces
The director can choose the three jazz pieces and none of the seven non-jazz pieces in only one way: ${}_3C_3 \times {}_7C_0 = 1$.

The total number of combinations with at least one jazz piece is $63 + 21 + 1 = 85$.

Solution 2 Indirect Method

You can find the total number of possible combinations of three pieces of music and subtract those that do not have any jazz.

The total number of ways of choosing any three pieces from the ten available is ${}_{10}C_3 = 120$. The number of ways of not picking any jazz, that is, choosing only from the non-jazz pieces is ${}_7C_3 = 35$.

Thus, the number of ways of choosing at least one jazz piece is $120 - 35 = 85$.

Here is a summary of ways to approach questions involving choosing or selecting objects.

Is order important?

Yes: Use permutations. Can the same objects be selected more than once (like digits for a telephone number)?
- **Yes:** Use the fundamental counting principle.
- **No:** Are some of the objects identical?
  - **Yes:** Use the formula $\dfrac{n!}{a!b!c!\ldots}$.
  - **No:** Use ${}_nP_r = \dfrac{n!}{(n-r)!}$.

No: Use combinations. Are you choosing exactly r objects?
- **Yes:** Could some of the objects be identical?
  - **Yes:** Count the individual cases.
  - **No:** Use ${}_nC_r = \dfrac{n!}{(n-r)!r!}$
- **No:** Are some of the objects identical?
  - **Yes:** Use $(p+1)(q+1)(r+1) - 1$ to find the total number of combinations with at least one object.
  - **No:** Use 2^n to find the total number of combinations; subtract 1 if you do not want to include the null set.

Key Concepts

- Use the formula $(p + 1)(q + 1)(r + 1) \ldots - 1$ to find the total number of selections of at least one item that can be made from p items of one kind, q of a second kind, r of a third kind, and so on.
- A set with n distinct elements has 2^n subsets including the null set.
- For combinations with some identical elements, you often have to consider all possible cases individually.
- In a situation where you must choose *at least* one particular item, either consider the total number of choices available minus the number without the desired item or add all the cases in which it is possible to have the desired item.

Communicate Your Understanding

1. Give an example of a situation where you would use the formula $(p + 1)(q + 1)(r + 1) \ldots - 1$. Explain why this formula applies.
2. Give an example of a situation in which you would use the expression $2^n - 1$. Explain your reasoning.
3. Using examples, describe two different ways to solve a problem where *at least* one particular item must be chosen. Explain why both methods give the same answer.

Practise

A

1. How many different sums of money can you make with a penny, a dime, a one-dollar coin, and a two-dollar coin?
2. How many different sums of money can be made with one \$5 bill, two \$10 bills, and one \$50 bill?
3. How many subsets are there for a set with
   a) two distinct elements?
   b) four distinct elements?
   c) seven distinct elements?
4. In how many ways can a committee with eight members form a subcommittee with at least one person on it?

B

5. Determine whether the following questions involve permutations or combinations and list any formulas that would apply.
   a) How many committees of 3 students can be formed from 12 students?
   b) In how many ways can 12 runners finish first, second, and third in a race?
   c) How many outfits can you assemble from three pairs of pants, four shirts, and two pairs of shoes?
   d) How many two-letter arrangements can be formed from the word *star*?

Apply, Solve, Communicate

6. Seven managers and eight sales representatives volunteer to attend a trade show. Their company can afford to send five people. In how many ways could they be selected
   a) without any restriction?
   b) if there must be at least one manager and one sales representative chosen?

7. **Application** A cookie jar contains three chocolate-chip, two peanut-butter, one lemon, one almond, and five raisin cookies.
   a) In how many ways can you reach into the jar and select some cookies?
   b) In how many ways can you select some cookies, if you must include at least one chocolate-chip cookie?

8. A project team of 6 students is to be selected from a class of 30.
   a) How many different teams can be selected?
   b) Pierre, Gregory, and Miguel are students in this class. How many of the teams would include these 3 students?
   c) How many teams would not include Pierre, Gregory, and Miguel?

9. The game of euchre uses only the 9s, 10s, jacks, queens, kings, and aces from a standard deck of cards. How many five-card hands have
   a) all red cards?
   b) at least two red cards?
   c) at most two red cards?

10. If you are dealing from a standard deck of 52 cards,
    a) how many different 4-card hands could have at least one card from each suit?
    b) how many different 5-card hands could have at least one spade?
    c) how many different 5-card hands could have at least two face cards (jacks, queens, or kings?

11. The number 5880 can be factored into prime divisors as $2 \times 2 \times 2 \times 3 \times 5 \times 7 \times 7$.
    a) Determine the total number of divisors of 5880.
    b) How many of the divisors are even?
    c) How many of the divisors are odd?

12. **Application** A theme park has a variety of rides. There are seven roller coasters, four water rides, and nine story rides. If Stephanie wants to try one of each type of ride, how many different combinations of rides could she choose?

13. Shuwei finds 11 shirts in his size at a clearance sale. How many different purchases could Shuwei make?

14. **Communication** Using the summary on page 285, draw a flow chart for solving counting problems.

15. a) How many different teams of 4 students could be chosen from the 15 students in the grade-12 Mathematics League?
    b) How many of the possible teams would include the youngest student in the league?
    c) How many of the possible teams would exclude the youngest student?

16. **Inquiry/Problem Solving**
    a) Use combinations to determine how many diagonals there are in
       i) a pentagon ii) a hexagon
    b) Draw sketches to verify your answers in part a).

17. A school is trying to decide on new school colours. The students can choose three colours from gold, black, green, blue, red, and white, but they know that another school has already chosen black, gold, and red. How many different combinations of three colours can the students choose?

18. **Application** The social convenor has 12 volunteers to work at a school dance. Each dance requires 2 volunteers at the door, 4 volunteers on the floor, and 6 floaters. Joe and Jim have not volunteered before, so the social convenor does not want to assign them to work together. In how many ways can the volunteers be assigned?

19. Jeffrey is a DJ at a local radio station. For the second hour of his shift, he must choose all his music from the top 100 songs for the week. Jeffery will play exactly 12 songs during this hour.

Chapter Problem

a) How many different stacks of discs could Jeffrey pull from the station's collection if he chooses at least 10 songs that are in positions 15 to 40 on the charts?

b) Jeffrey wants to start his second hour with a hard-rock song and finish with a pop classic. How many different play lists can Jeffrey prepare if he has chosen 4 hard rock songs, 5 soul pieces, and 3 pop classics?

c) Jeffrey has 8 favourite songs currently on the top 100 list. How many different subsets of these songs could he choose to play during his shift?

ACHIEVEMENT CHECK

Knowledge / Understanding	Thinking / Inquiry / Problem Solving	Communication	Application

20. There are 52 white keys on a piano. The lowest key is A. The keys are designated A, B, C, D, E, F, and G in succession, and then the sequence of letters repeats, ending with a C for the highest key.

a) If five notes are played simultaneously, in how many ways could the notes all be

i) As? **ii)** Gs?

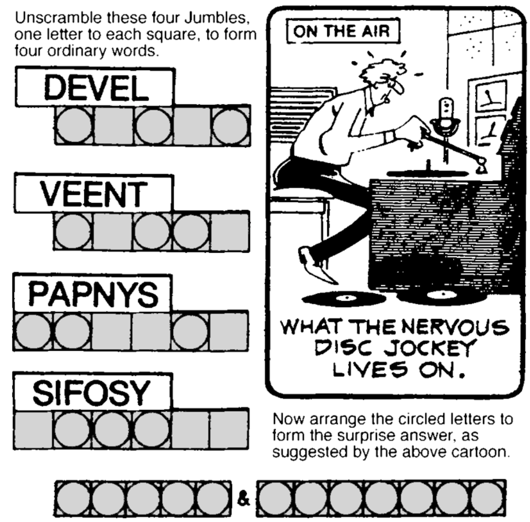
iii) the same letter? **iv)** different letters?

b) If the five keys are played in order, how would your answers in part a) change?

21. **Communication**

a) How many possible combinations are there for the letters in the three circles for each of the clue words in this puzzle?

b) Explain why you cannot answer part a) with a single ${}_nC_r$ calculation for each word.



22. Determine the number of ways of selecting four letters, without regard for order, from the word *parallelogram*.

C

23. **Inquiry / Problem Solving** Suppose the artist in Example 1 of this section had two apples, two oranges, and two pears in his refrigerator. How many combinations does he have to choose from if he wants to paint a still-life with

a) two pieces of fruit?

b) three pieces of fruit?

c) four pieces of fruit?

24. How many different sums of money can be formed from one \$2 bill, three \$5 bills, two \$10 bills, and one \$20 bill?