6.1 Basic Probability Concepts

How likely is rain tomorrow? What are the chances that you will pass your driving test on the first attempt? What are the odds that the flight will be on time when you go to meet someone at the airport?

Probability is the branch of mathematics that attempts to predict answers to questions like these. As the word *probability* suggests, you can often predict only what *might* happen. However, you may be able to calculate how likely it is. For example, if the weather report forecasts a 90% chance of rain, there is still that slight possibility that sunny skies will prevail. While there are no sure answers, in this case it *probably* will rain.

INVESTIGATE & INQUIRE: A Number Game

Work with a partner. Have each partner take three identical slips of paper, number them 1, 2, and 3, and place them in a hat, bag, or other container. For each trial, both partners will randomly select one of their three slips of paper. Replace the slips after each trial. Score points as follows:

- If the product of the two numbers shown is less than the sum, Player A gets a point.
- If the product is greater than the sum, Player B gets a point.
- If the product and sum are equal, neither player gets a point.

1. Predict who has the advantage in this game. Explain why you think so.

2. Decide who will be Player A by flipping a coin or using the random number generator on a graphing calculator. Organize your results in a table like the one below.

Trial	1	2	3	4	5	6	7	8	9	10
Number drawn by A										
Number drawn by B										
Product										
Sum										
Point awarded to:										

3. a) Record the results for 10 trials. Total the points and determine the winner. Do the results confirm your prediction? Have you changed your opinion on who has the advantage? Explain.
   b) To estimate a probability for each player getting a point, divide the number of points each player earned by the total number of trials.

4. a) Perform 10 additional trials and record point totals for each player over all 20 trials. Estimate the probabilities for each player, as before.
   b) Are the results for 20 trials consistent with the results for 10 trials? Explain.
   c) Are your results consistent with those of your classmates? Comment on your findings.

5. Based on your results for 20 trials, predict how many points each player will have after 50 trials.

6. Describe how you could alter the game so that the other player has the advantage.

The investigation you have just completed is an example of a **probability experiment**. In probability, an experiment is a well-defined process consisting of a number of **trials** in which clearly distinguishable **outcomes**, or possible results, are observed.

The **sample space, *S*,** of an experiment is the set of all possible outcomes. For the sum/product game in the investigation, the outcomes are all the possible pairings of slips drawn by the two players. For example, if Player A draws 1 and Player B draws 2, you can label this outcome (1, 2). In this particular game, the result is the same for the outcomes (1, 2) and (2, 1), but with different rules it might be important who draws which number, so it makes sense to view the two outcomes as different.

Outcomes are often equally likely. In the sum/product game, each possible pairing of numbers is as likely as any other. Outcomes are often grouped into **events**. An example of an event is drawing slips for which the product is greater than the sum, and there are several outcomes in which this event happens. Different events often have different chances of occurring. Events are usually labelled with capital letters.

Example 1 Outcomes and Events

Let event A be a point awarded to Player A in the sum/product game. List the outcomes that make up event A.

Solution

Player A earns a point if the sum of the two numbers is greater than the product. This event is sometimes written as event $A = \{\text{sum} > \text{product}\}$. A useful technique in probability is to tabulate the possible outcomes.

Sums

		Player A		
		1	2	3
Player B	1	2	3	4
	2	3	4	5
	3	4	5	6

Products

		Player A		
		1	2	3
Player B	1	1	2	3
	2	2	4	6
	3	3	6	9

Use the tables shown to list the outcomes where the sum is greater than the product:
(1, 1), (1, 2), (1, 3), (2, 1), (3, 1)
These outcomes make up event A. Using this list, you can also write event A as event $A = \{(1, 1), (1, 2), (1, 3), (2, 1), (3, 1)\}$

The **probability** of event A, $\boldsymbol{P(A)}$, is a quantified measure of the likelihood that the event will occur. *The probability of an event is always a value between 0 and 1.* A probability of 0 indicates that the event is impossible, and 1 signifies that the event is a certainty. Most events in probability studies fall somewhere between these extreme values. Probabilities less than 0 or greater than 1 have no meaning. Probability can be expressed as fractions, decimals, or percents. Probabilities expressed as percents are always between 0% and 100%. For example, a 70% chance of rain tomorrow means the same as a probability of 0.7, or $\frac{7}{10}$, that it will rain.

The three basic types of probability are

- empirical probability, based on direct observation or experiment
- theoretical probability, based on mathematical analysis
- subjective probability, based on informed guesswork

The **empirical probability** of a particular event (also called **experimental** or **relative frequency probability**) is determined by dividing the number of times that the event actually occurs in an experiment by the number of trials. In the sum/product investigation, you were calculating empirical probabilities. For example, if you had found that in the first ten trials, the product was greater than the sum four times, then the empirical probability of this event would be

$$\begin{aligned} P(A) &= \frac{4}{10} \\ &= \frac{2}{5} \text{ or } 0.4 \end{aligned}$$

The **theoretical probability** of a particular event is deduced from analysis of the possible outcomes. Theoretical probability is also called **classical** or ***a priori*** probability. *A priori* is Latin for "from the preceding," meaning based on analysis rather than experiment.

For example, if all possible outcomes are equally likely, then

$$P(A) = \frac{n(A)}{n(S)}$$

where $n(A)$ is the number of outcomes in which event A can occur, and $n(S)$ is the total number of possible outcomes. You used tables to list the outcomes for A in Example 1, and this technique allows you to find the theoretical probability $P(A)$ by counting $n(A) = 5$ and $n(S) = 9$. Another way to determine the values of $n(A)$ and $n(S)$ is by organizing the information in a tree diagram.

Project Prep

You will need to determine theoretical probabilities to design and analyse your game in the probability project.

Example 2 Using a Tree Diagram to Calculate Probability

Determine the theoretical probabilities for each key event in the sum/product game.

Solution

The tree diagram shows the nine possible outcomes, each equally likely, for the sum/product game.

		product		sum
1	1	1	<	2
	2	2	<	3
	3	3	<	4
2	1	2	<	3
	2	4	=	4
	3	6	>	5
3	1	3	<	4
	2	6	>	5
	3	9	>	6

Let event A be a point for Player A, event B a point for Player B, and event C a tie between sum and product. From the tree diagram, five of the nine possible outcomes have the sum greater than the product. Therefore, the theoretical probability of this event is

$$\begin{aligned} P(A) &= \frac{n(A)}{n(S)} \\ &= \frac{5}{9} \end{aligned}$$

Similarly,

$$\begin{aligned} P(B) &= \frac{n(B)}{n(S)} \\ &= \frac{3}{9} \end{aligned} \quad \text{and} \quad \begin{aligned} P(C) &= \frac{n(C)}{n(S)} \\ &= \frac{1}{9} \end{aligned}$$

In Example 2, you know that one, and only one, of the three events will occur. The sum of the probabilities of all possible events always equals 1.

$$\begin{aligned} P(A) + P(B) + P(C) &= \frac{5}{9} + \frac{3}{9} + \frac{1}{9} \\ &= 1 \end{aligned}$$

Here, the numerator in each fraction represents the number of ways that each event can occur. The total of these numerators is the total number of possible outcomes, which is equal to the denominator.

Empirical probabilities may differ sharply from theoretical probabilities when only a few trials are made. Such **statistical fluctuation** can result in an event occurring more frequently or less frequently than theoretical probability suggests. Over a large number of trials, however, statistical fluctuations tend to cancel each other out, and empirical probabilities usually approach theoretical values. Statistical fluctuations often appear in sports, for example, where a team can enjoy a temporary winning streak that is not sustainable over an entire season.

In most problems, you will be determining theoretical probability. Therefore, from now on you may take the term *probability* to mean *theoretical probability* unless stated otherwise.

Example 3 Dice Probabilities

Many board games involve a roll of two six-sided dice to see how far you may move your pieces or counters. What is the probability of rolling a total of 7?

Solution

The table shows the totals for all possible rolls of two dice.

		First Die					
		1	2	3	4	5	6
Second Die	1	2	3	4	5	6	7
	2	3	4	5	6	7	8
	3	4	5	6	7	8	9
	4	5	6	7	8	9	10
	5	6	7	8	9	10	11
	6	7	8	9	10	11	12

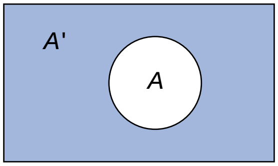
To calculate the probability of a particular total, count the number of times it appears in the table. For event A = {rolling 7},

$$
\begin{aligned}
P(A) &= \frac{n(A)}{n(S)} \\
&= \frac{n(\text{rolls totalling 7})}{n(\text{all possible rolls})} \\
&= \frac{6}{36} \\
&= \frac{1}{6}
\end{aligned}
$$

The probability of rolling a total of 7 is $\frac{1}{6}$.

A useful and important concept in probability is the complement of an event. The **complement** of event A, $\boldsymbol{A'}$ or **~*A***, is the event that "event A does *not* happen." Thus, whichever outcomes make up A, all the other outcomes make up A'. Because A and A' together include all possible outcomes, the sum of their probabilities must be 1. Thus,

$$P(A) + P(A') = 1 \qquad \text{and} \qquad P(A') = 1 - P(A)$$



The event A' is usually called "A-prime," or sometimes "not-A"; ~A is called "tilde-A."

Example 4 The Complement of an Event

What is the probability that a randomly drawn integer between 1 and 40 is *not* a perfect square?

Solution

Let event A = {a perfect square}. Then, the complement of A is the event A' = {*not* a perfect square}. In this case, you need to calculate $P(A')$, but it is easier to do this by finding $P(A)$ first. There are six perfect squares between 1 and 40: 1, 4, 9, 16, 25, and 36. The probability of a perfect square is, therefore,

$$\begin{aligned} P(A) &= \frac{n(A)}{n(S)} \\ &= \frac{6}{40} \\ &= \frac{3}{20} \end{aligned}$$

Thus,

$$\begin{aligned} P(A') &= 1 - P(A) \\ &= 1 - \frac{3}{20} \\ &= \frac{17}{20} \end{aligned}$$

There is a $\frac{17}{20}$ or 85% chance that a random integer between 1 and 40 will not be a perfect square.

Subjective probability, the third basic type of probability, is an estimate of likelihood based on intuition and experience—an educated guess. For example, a well-prepared student may be 90% confident of passing the next data management test. Subjective probabilities often figure in everyday speech in expressions such as "I think the team has only a 10% chance of making the finals this year."

Example 5 Determining Subjective Probability

Estimate the probability that

a) the next pair of shoes you buy will be the same size as the last pair you bought

b) an expansion baseball team will win the World Series in their first season

c) the next person to enter a certain coffee shop will be male

Solution

a) There is a small chance that the size of your feet has changed significantly or that different styles of shoes may fit you differently, so 80–90% would be a reasonable subjective probability that your next pair of shoes will be the same size as your last pair.

b) Expansion teams rarely do well during their first season, and even strong teams have difficulty winning the World Series. The subjective probability of a brand-new team winning the World Series is close to zero.

c) Without more information about the coffee shop in question, your best estimate is to assume that the shop's patrons are representative of the general population. This assumption gives a subjective probability of 50% that the next customer will be male.

WEB CONNECTION

www.mcgrawhill.ca/links/MDM12

For some interesting baseball statistics, visit the above web site and follow the links. Write a problem that could be solved using probabilities.

Note that the answers in Example 5 contain estimates, assumptions, and, in some cases, probability *ranges*. While not as rigorous a measure as theoretical or empirical probability, subjective probabilities based on educated guesswork can still prove useful in some situations.

Key Concepts

- A probability experiment is a well-defined process in which clearly identifiable outcomes are measured for each trial.
- An event is a collection of outcomes satisfying a particular condition. The probability of an event can range between 0 (impossible) and 1 or 100% (certain).
- The empirical probability of an event is the number of times the event occurs divided by the total number of trials.
- The theoretical probability of an event A is given by $P(A) = \frac{n(A)}{n(S)}$, where $n(A)$ is the number of outcomes making up A, $n(S)$ is the total number of outcomes in the sample space S, and all outcomes are equally likely to occur.
- A subjective probability is based on intuition and previous experience.
- If the probability of event A is given by $P(A)$, then the probability of the complement of A is given by $P(A') = 1 - P(A)$.

Communicate Your Understanding

1. Give two synonyms for the word *probability*.

2. **a)** Explain why $P(A) + P(A') = 1$.

b) Explain why probabilities less than 0 or greater than 1 have no meaning.

3. Explain the difference between theoretical, empirical, and subjective probability. Give an example of how you would determine each type.

4. Describe three situations in which statistical fluctuations occur.

5. **a)** Describe a situation in which you might determine the probability of event A indirectly by calculating $P(A')$ first.

b) Will this method always yield the same result as calculating $P(A)$ directly?

c) Defend your answer to part b) using an explanation or proof, supported by an example.

Practise

A

1. Determine the probability of

a) tossing heads with a single coin

b) tossing two heads with two coins

c) tossing at least one head with three coins

d) rolling a composite number with one die

e) not rolling a perfect square with two dice

f) drawing a face card from a standard deck of cards

2. Estimate a subjective probability of each of the following events. Provide a rationale for each estimate.

a) the sun rising tomorrow

b) it never raining again

c) your passing this course

d) your getting the next job you apply for

3. Recall the sum/product game at the beginning of this section. Suppose that the game were altered so that the slips of paper showed the numbers 2, 3, and 4, instead of 1, 2, and 3.

a) Identify all the outcomes that will produce each of the three possible events

i) $p > s$ **ii)** $p < s$ **iii)** $p = s$

b) Which player has the advantage in this situation?

Apply, Solve, Communicate

4. The town planning department surveyed residents of a town about home ownership. The table shows the results of the survey.

Residents	At Address Less Than 2 Years	At Address More Than 2 Years	Total for Category
Owners	2000	8000	10 000
Renters	4500	1500	6 000
Total	6500	9500	16 000

Determine the following probabilities.

a) P(resident owns home)

b) P(resident rents and has lived at present address less than two years)

c) P(homeowner has lived at present address more than two years)

B

5. **Application** Suppose your school's basketball team is playing a four-game series against another school. So far this season, each team has won three of the six games in which they faced each other.

a) Draw a tree diagram to illustrate all possible outcomes of the series.

b) Use your tree diagram to determine the probability of your school winning exactly two games.

c) What is the probability of your school sweeping the series (winning all four games)?

d) Discuss any assumptions you made in the calculations in parts b) and c).

6. **Application** Suppose that a graphing calculator is programmed to generate a random natural number between 1 and 10 inclusive. What is the probability that the number will be prime?

7. **Communication**

a) A game involves rolling two dice. Player A wins if the throw totals 5, 7, or 9. Player B wins if any other total is thrown. Which player has the advantage? Explain.

b) Suppose the game is changed so that Player A wins if 5, 7, or doubles (both dice showing the same number) are thrown. Who has the advantage now? Explain.

c) Design a similar game in which each player has an equal chance of winning.

8. a) Based on the randomly tagged sample, what is the empirical probability that a deer captured at random will be a doe?

b) If ten deer are captured at random, how many would you expect to be bucks?

C

9. **Inquiry/Problem Solving** Refer to the prime number experiment in question 6. What happens to the probability if you change the upper limit of the sample space? Use a graphing calculator or appropriate computer software to investigate this problem. Let A be the event that the random natural number will be a prime number. Let the random number be between 1 and n inclusive. Predict what you think will happen to $P(A)$ as n increases. Investigate $P(A)$ as a function of n, and reflect on your hypothesis. Did you observe what you expected? Why or why not?

10. Suppose that the Toronto Blue Jays face the New York Yankees in the division final. In this best-of-five series, the winner is the first team to win three games. The games are played in Toronto and in New York, with Toronto hosting the first, second, and if needed, fifth games. The consensus among experts is that Toronto has a 65% chance of winning at home and a 40% chance of winning in New York.

a) Construct a tree diagram to illustrate all the possible outcomes.

b) What is the chance of Toronto winning in three straight games?

c) For each outcome, add to your tree diagram the probability of that outcome.

d) **Communication** Explain how you found your answers to parts b) and c).

11. **Communication** Prior to a municipal election, a public-opinion poll determined that the probability of each of the four candidates winning was as follows:

Jonsson 10%

Trimble 32%

Yakamoto 21%

Audette 37%

a) How will these probabilities change if Jonsson withdraws from the race after ballots are cast?

b) How will these probabilities change if Jonsson withdraws from the race before ballots are cast?

c) Explain why your answers to a) and b) are different.

12. **Inquiry/Problem Solving** It is known from studying past tests that the correct answers to a certain university professor's multiple-choice tests exhibit the following pattern.

Correct Answer	Percent of Questions
A	15%
B	25%
C	30%
D	15%
E	15%

a) Devise a strategy for guessing that would maximize a student's chances for success, assuming that the student has no idea of the correct answers. Explain your method.

b) Suppose that the study of past tests revealed that the correct answer choice for any given question was the same as that of the immediately preceding question only 10% of the time. How would you use this information to adjust your strategy in part a)? Explain your reasoning.