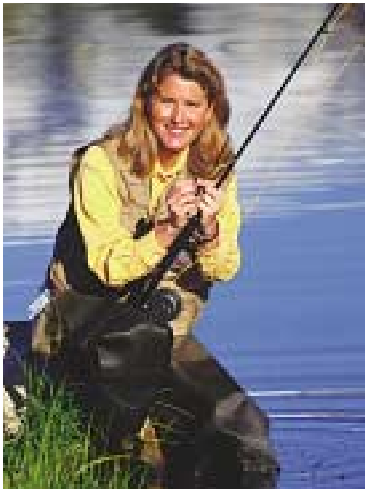
6.3 Probabilities Using Counting Techniques

How likely is it that, in a game of cards, you will be dealt just the hand that you need? Most card players accept this question as an unknown, enjoying the unpredictability of the game, but it can also be interesting to apply counting analysis to such problems.

In some situations, the possible outcomes are not easy or convenient to count individually. In many such cases, the counting techniques of permutations and combinations (see Chapters 4 and 5, respectively) can be helpful for calculating theoretical probabilities, or you can use a simulation to determine an empirical probability.

INVESTIGATE & INQUIRE: Fishing Simulation



Suppose a pond has only three types of fish: catfish, trout, and bass, in the ratio 5:2:3. There are 50 fish in total. Assuming you are allowed to catch only three fish before throwing them back, consider the following two events:

- event A = {catching three trout}
- event B = {catching the three types of fish, in alphabetical order}

1. Carry out the following probability experiment, independently or with a partner. You can use a hat or paper bag to represent the pond, and some differently coloured chips or markers to represent the fish. How many of each type of fish should you release into the pond? Count out the appropriate numbers and shake the container to simulate the fish swimming around.

2. Draw a tree diagram to illustrate the different possible outcomes of this experiment.

3. Catch three fish, one at a time, and record the results in a table. Replace all three fish and shake the container enough to ensure that they are randomly distributed. Repeat this process for a total of ten trials.

4. Based on these ten trials, determine the empirical probability of event A, catching three trout. How accurate do you think this value is? Compare your results with those of the rest of the class. How can you obtain a more accurate empirical probability?

5. Repeat step 4 for event B, which is to catch a bass, catfish, and trout in order.

6. Perform step 3 again for 10 new trials. Calculate the empirical probabilities of events A and B, based on your 20 trials. Do you think these probabilities are more accurate than those from 10 trials? Explain why or why not.

7. If you were to repeat the experiment for 50 or 100 trials, would your results be more accurate? Why or why not?

8. In this investigation, you knew exactly how many of each type of fish were in the pond because they were counted out at the beginning. Describe how you could use the techniques of this investigation to estimate the ratios of different species in a real pond.

This section examines methods for determining the theoretical probabilities of successive or multiple events.

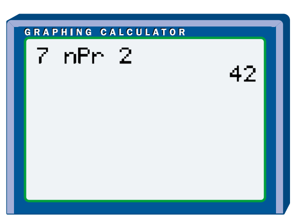
Example 1 Using Permutations

Two brothers enter a race with five friends. The racers draw lots to determine their starting positions. What is the probability that the older brother will start in lane 1 with his brother beside him in lane 2?

Solution

A permutation ${}_nP_r$, or $P(n, r)$, is the number of ways to select r objects from a set of n objects, *in a certain order*. (See Chapter 4 for more about permutations.) The sample space is the total number of ways the first two lanes can be occupied. Thus,

$$\begin{aligned} n(S) &= {}_7P_2 \\ &= \frac{7!}{(7-2)!} \\ &= \frac{7!}{5!} \\ &= \frac{7 \times 6 \times (5!)}{5!} \\ &= 42 \end{aligned}$$



The specific outcome of the older brother starting in lane 1 and the younger brother starting in lane 2 can only happen one way, so $n(A) = 1$. Therefore,

$$\begin{aligned} P(A) &= \frac{n(A)}{n(S)} \\ &= \frac{1}{42} \end{aligned}$$

The probability that the older brother will start in lane 1 next to his brother in lane 2 is $\frac{1}{42}$, or approximately 2.3%.

Example 2 Probability Using Combinations

A focus group of three members is to be randomly selected from a medical team consisting of five doctors and seven technicians.

a) What is the probability that the focus group will be comprised of doctors only?

b) What is the probability that the focus group will not be comprised of doctors only?

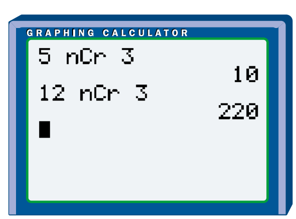
Solution

a) A combination ${}_nC_r$, also written $C(n, r)$ or $\binom{n}{r}$, is the number of ways to select r objects from a set of n objects, *in any order*. (See Chapter 5 for more about combinations.) Let event A be selecting three doctors to form the focus group. The number of possible ways to make this selection is

$$\begin{aligned} n(A) &= {}_5C_3 \\ &= \frac{5!}{3!(5-3)!} \\ &= \frac{5 \times 4 \times 3!}{3! \times 2!} \\ &= \frac{20}{2} \\ &= 10 \end{aligned}$$

However, the focus group can consist of any three people from the team of 12.

$$\begin{aligned} n(S) &= {}_{12}C_3 \\ &= \frac{12!}{3!(12-3)!} \\ &= \frac{12 \times 11 \times 10 \times 9!}{3! \times 9!} \\ &= \frac{1320}{6} \\ &= 220 \end{aligned}$$



The probability of selecting a focus group of doctors only is

$$\begin{aligned} P(A) &= \frac{n(A)}{n(S)} \\ &= \frac{10}{220} \\ &= \frac{1}{22} \end{aligned}$$

The probability of selecting a focus group consisting of three doctors is $\frac{1}{22}$, or approximately 0.045.

b) Either the focus group is comprised of doctors only, or it is not. Therefore, the probability of the complement of A, $P(A')$, gives the desired result.

$$\begin{aligned} P(A') &= 1 - P(A) \\ &= 1 - \frac{1}{22} \\ &= \frac{21}{22} \end{aligned}$$

So, the probability of selecting a focus group not comprised of doctors only is $\frac{21}{22}$, or approximately 0.955.

Project Prep

When you determine the classical probabilities for your probability project, you may need to apply the counting techniques of permutations and combinations.

Example 3 Probability Using the Fundamental Counting Principle

What is the probability that two or more students out of a class of 24 will have the same birthday? Assume that no students were born on February 29.

Solution 1 Using Pencil and Paper

The simplest method is to find the probability of the complementary event that no two people in the class have the same birthday.

Pick two students at random. The second student has a different birthday than the first for 364 of the 365 possible birthdays. Thus, the probability that the two students have different birthdays is $\frac{364}{365}$. Now add a third student. Since there are 363 ways this person can have a different birthday from the other two students, the probability that all three students have different birthdays is $\frac{364}{365} \times \frac{363}{365}$. Continuing this process, the probability that none of the 24 people have the same birthday is

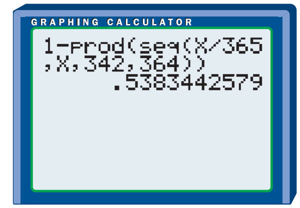
$$\begin{aligned} P(A') &= \frac{n(A')}{n(S)} \\ &= \frac{364}{365} \times \frac{363}{365} \times \frac{362}{365} \times \ldots \times \frac{342}{365} \\ &\doteq 0.462 \end{aligned}$$

$$\begin{aligned} P(A) &= 1 - P(A') \\ &= 1 - 0.462 \\ &= 0.538 \end{aligned}$$

The probability that at least two people in the group have the same birthday is approximately 0.538.

Solution 2 Using a Graphing Calculator

Use the iterative functions of a graphing calculator to evaluate the formula above much more easily. The **prod(function** on the LIST MATH menu will find the product of a series of numbers. The **seq(function** on the LIST OPS menu generates a sequence for the range you specify. Combining these two functions allows you to calculate the probability in a single step.



Key Concepts

- In probability experiments with many possible outcomes, you can apply the fundamental counting principle and techniques using permutations and combinations.
- Permutations are useful when order is important in the outcomes; combinations are useful when order is not important.

Communicate Your Understanding

1. In the game of bridge, each player is dealt 13 cards out of the deck of 52. Explain how you would determine the probability of a player receiving
   a) all hearts **b)** all hearts in ascending order
2. **a)** When should you apply permutations in solving probability problems, and when should you apply combinations?
   b) Provide an example of a situation where you would apply permutations to solve a probability problem, other than those in this section.
   c) Provide an example of a situation where you would apply combinations to solve a probability problem, other than those in this section.

Practise

A

1. Four friends, two females and two males, are playing contract bridge. Partners are randomly assigned for each game. What is the probability that the two females will be partners for the first game?
2. What is the probability that at least two out of a group of eight friends will have the same birthday?
3. A fruit basket contains five red apples and three green apples. Without looking, you randomly select two apples. What is the probability that
   a) you will select two red apples?
   b) you will not select two green apples?
4. Refer to Example 1. What is the probability that the two brothers will start beside each other in any pair of lanes?

Apply, Solve, Communicate

B

5. An athletic committee with three members is to be randomly selected from a group of six gymnasts, four weightlifters, and eight long-distance runners. Determine the probability that

a) the committee is comprised entirely of runners

b) the committee is represented by each of the three types of athletes

6. A messy drawer contains three black socks, five blue socks, and eight white socks, none of which are paired up. If the owner grabs two socks without looking, what is the probability that both will be white?

7. a) A family of nine has a tradition of drawing two names from a hat to see whom they will each buy presents for. If there are three sisters in the family, and the youngest sister is always allowed the first draw, determine the probability that the youngest sister will draw both of the other two sisters' names. If she draws her own name, she replaces it and draws another.

b) Suppose that the tradition is modified one year, so that the first person whose name is drawn is to receive a "main" present, and the second a less expensive, "fun" present. Determine the probability that the youngest sister will give a main present to the middle sister and a fun present to the eldest sister.

8. Application

a) Laura, Dave, Monique, Marcus, and Sarah are going to a party. What is the probability that two of the girls will arrive first?

b) What is the probability that the friends will arrive in order of ascending age?

c) What assumptions must be made in parts a) and b)?

9. A hockey team has two goalies, six defenders, eight wingers, and four centres. If the team randomly selects four players to attend a charity function, what is the likelihood that

a) they are all wingers?

b) no goalies or centres are selected?

10. Application A lottery promises to award ten grand-prize trips to Hawaii and sells 5 400 000 tickets.

a) Determine the probability of winning a grand prize if you buy

i) 1 ticket

ii) 10 tickets

iii) 100 tickets

b) Communication How many tickets do you need to buy in order to have a 5% chance of winning a grand prize? Do you think this strategy is sensible? Why or why not?

c) How many tickets do you need to ensure a 50% chance of winning?

11. Suki is enrolled in one data-management class at her school and Leo is in another. A school quiz team will have four volunteers, two randomly selected from each of the two classes. Suki is one of five volunteers from her class, and Leo is one of four volunteers from his. Calculate the probability of the two being on the team and explain the steps in your calculation.

12. a) Suppose 4 of the 22 tagged bucks are randomly chosen for a behaviour study. What is the probability that

i) all four bucks have the cross-hatched antlers?

ii) at least one buck has cross-hatched antlers?

b) If two of the seven cross-hatched males are randomly selected for a health study, what is the probability that the eldest of the seven will be selected first, followed by the second eldest?

ACHIEVEMENT CHECK

Knowledge / Understanding	Thinking / Inquiry / Problem Solving	Communication	Application

13. Suppose a bag contains the letters to spell *probability*.

a) How many four-letter arrangements are possible using these letters?

b) What is the probability that Barb chooses four letters from the bag in the order that spell her name?

c) Pick another four-letter arrangement and calculate the probability that it is chosen.

d) What four-letter arrangement would be most likely to be picked? Explain your reasoning.

C

14. Communication Refer to the fishing investigation at the beginning of this section.

a) Determine the theoretical probability of

i) catching three trout

ii) catching a bass, catfish, and trout in alphabetical order

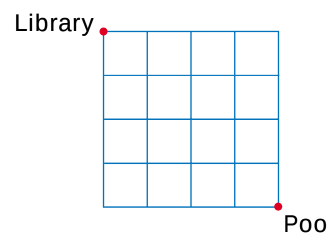
b) How do these results compare with the empirical probabilities from the investigation? How do you account for any differences?

c) Could the random-number generator of a graphing calculator be used to simulate this investigation? If so, explain how. If not, explain why.

d) Outline the steps you would use to model this problem with software such as Fathom™ or a spreadsheet.

e) Is the assumption that the fish are randomly distributed likely to be completely correct? Explain. What other assumptions might affect the accuracy of the calculated probabilities?

15. A network of city streets forms square blocks as shown in the diagram.



Jeanine leaves the library and walks toward the pool at the same time as Miguel leaves the pool and walks toward the library. Neither person follows a particular route, except that both are always moving toward their destination. What is the probability that they will meet if they both walk at the same rate?

16. Inquiry / Problem Solving A committee is formed by randomly selecting from eight nurses and two doctors. What is the minimum committee size that ensures at least a 90% probability that it will not be comprised of nurses only?