6.4 Dependent and Independent Events

If you have two examinations next Tuesday, what is the probability that you will pass both of them? How can you predict the risk that a critical network server and its backup will both fail? If you flip an ordinary coin repeatedly and get heads 99 times in a row, is the next toss almost certain to come up tails?

In such situations, you are dealing with **compound events** involving two or more separate events.

INVESTIGATE & INQUIRE: Getting Out of Jail in MONOPOLY®

While playing MONOPOLY® for the first time, Kenny finds himself in jail. To get out of jail, he needs to roll doubles on a pair of standard dice.

1. Determine the probability that Kenny will roll doubles on his first try.
2. Suppose that Kenny fails to roll doubles on his first two turns in jail. He reasons that on his next turn, his odds are now 50/50 that he will get out of jail. Explain how Kenny has reasoned this.
3. Do you agree or disagree with Kenny's reasoning? Explain.
4. What is the probability that Kenny will get out of jail on his third attempt?
5. After how many turns is Kenny certain to roll doubles? Explain.
6. Kenny's opponent, Roberta, explains to Kenny that each roll of the dice is an independent event and that, since the relatively low probability of rolling doubles never changes from trial to trial, Kenny may never get out of jail and may as well just forfeit the game. Explain the flaws in Roberta's analysis.

In some situations involving compound events, the occurrence of one event has no effect on the occurrence of another. In such cases, the events are **independent**.

Example 1 Simple Independent Events

a) A coin is flipped and turns up heads. What is the probability that the second flip will turn up heads?

b) A coin is flipped four times and turns up heads each time. What is the probability that the fifth trial will be heads?

c) Find the probability of tossing five heads in a row.

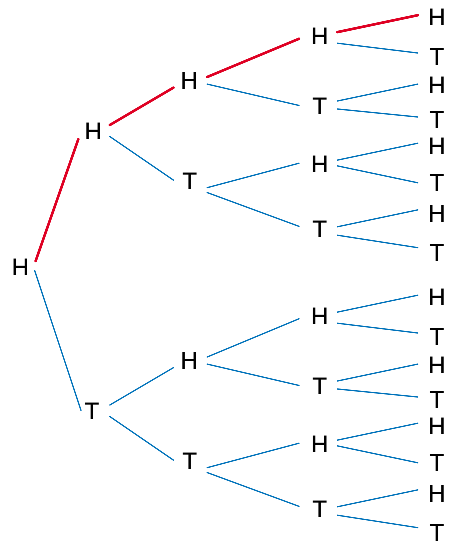
d) Comment on any difference between your answers to parts b) and c).

Solution

a) Because these events are independent, the outcome of the first toss has no effect on the outcome of the second toss. Therefore, the probability of tossing heads the second time is 0.5.

b) Although you might think "tails has to come up sometime," there is still a 50/50 chance on each independent toss. The coin has no memory of the past four trials! Therefore, the fifth toss still has just a 0.5 probability of coming up heads.

c) Construct a tree diagram to represent five tosses of the coin.



There is an equal number of outcomes in which the first flip turns up tails.

The number of outcomes doubles with each trial. After the fifth toss, there are 2^5 or 32 possible outcomes, only one of which is five heads in a row. So, the probability of five heads in a row, prior to any coin tosses, is $\frac{1}{32}$ or 0.031 25.

d) The probability in part c) is much less than in part b). In part b), you calculate only the probability for the fifth trial on its own. In part c), you are finding the probability that every one of five separate events actually happens.

Example 2 Probability of Two Different Independent Events

A coin is flipped while a die is rolled. What is the probability of flipping heads and rolling 5 in a single trial?

Solution

Here, two independent events occur in a single trial. Let A be the event of flipping heads, and B be the event of rolling 5. The notation $P(A \text{ and } B)$ represents the compound, or joint, probability that both events, A and B, will occur simultaneously. For independent events, the probabilities can simply be multiplied together.

$$\begin{aligned} P(A \text{ and } B) &= P(A) \times P(B) \\ &= \frac{1}{2} \times \frac{1}{6} \\ &= \frac{1}{12} \end{aligned}$$

The probability of simultaneously flipping heads while rolling 5 is $\frac{1}{12}$ or approximately 8.3%

In general, the compound probability of two independent events can be calculated using the **product rule for independent events**:

$$P(A \text{ and } B) = P(A) \times P(B)$$

From the example above, you can see that the product rule for independent events agrees with common sense. The product rule can also be derived mathematically from the fundamental counting principle (see Chapter 4).

Proof:

A and B are separate events and so they correspond to separate sample spaces, S_A and S_B.

Their probabilities are thus

$$P(A) = \frac{n(A)}{n(S_A)} \text{ and } P(B) = \frac{n(B)}{n(S_B)}.$$

Call the sample space for the compound event S, as usual.

You know that

$$P(A \text{ and } B) = \frac{n(A \text{ and } B)}{n(S)} \qquad \textbf{(1)}$$

Because A and B are independent, you can apply the fundamental counting principle to get an expression for $n(A \text{ and } B)$.

$$n(A \text{ and } B) = n(A) \times n(B) \qquad \textbf{(2)}$$

Similarly, you can also apply the fundamental counting principle to get an expression for $n(S)$.

$$n(S) = n(S_A) \times n(S_B) \qquad \textbf{(3)}$$

Substitute equations (2) and (3) into equation (1).

$$\begin{aligned} P(A \text{ and } B) &= \frac{n(A)n(B)}{n(S_A)n(S_B)} \\ &= \frac{n(A)}{n(S_A)} \times \frac{n(B)}{n(S_B)} \\ &= P(A) \times P(B) \end{aligned}$$

Example 3 Applying the Product Rule for Independent Events

Soo-Ling travels the same route to work every day. She has determined that there is a 0.7 probability that she will wait for at least one red light and that there is a 0.4 probability that she will hear her favourite new song on her way to work.

a) What is the probability that Soo-Ling will not have to wait at a red light and will hear her favourite song?

b) What are the odds in favour of Soo-Ling having to wait at a red light and not hearing her favourite song?

Solution

a) Let A be the event of Soo-Ling having to wait at a red light, and B be the event of hearing her favourite song. Assume A and B to be independent events. In this case, you are interested in the combination A' and B.

$$\begin{aligned} P(A' \text{ and } B) &= P(A') \times P(B) \\ &= (1 - P(A)) \times P(B) \\ &= (1 - 0.7) \times 0.4 \\ &= 0.12 \end{aligned}$$

There is a 12% chance that Soo-Ling will hear her favourite song and not have to wait at a red light on her way to work.

b)
$$\begin{aligned} P(A \text{ and } B') &= P(A) \times P(B') \\ &= P(A) \times (1 - P(B)) \\ &= 0.7 \times (1 - 0.4) \\ &= 0.42 \end{aligned}$$

The probability of Soo-Ling having to wait at a red light and not hearing her favourite song is 42%.

The odds in favour of this happening are

$$\begin{aligned} \text{odds in favour} &= \frac{P(A \text{ and } B')}{1 - P(A \text{ and } B')} \\ &= \frac{42\%}{100\% - 42\%} \\ &= \frac{42}{58} \\ &= \frac{21}{29} \end{aligned}$$

The odds in favour of Soo-Ling having to wait at a red light and not hearing her favourite song are 21:29.

In some cases, the probable outcome of an event, B, depends directly on the outcome of another event, A. When this happens, the events are said to be **dependent**. The **conditional probability** of B, $\boldsymbol{P(B \mid A)}$, is the probability that B occurs, *given* that A has already occurred.

Example 4 Probability of Two Dependent Events

A professional hockey team has eight wingers. Three of these wingers are 30-goal scorers, or "snipers." Every fall the team plays an exhibition match with the club's farm team. In order to make the match more interesting for the fans, the coaches agree to select two wingers at random from the pro team to play for the farm team. What is the probability that two snipers will play for the farm team?

Solution

Let A = {first winger is a sniper} and B = {second winger is a sniper}. Three of the eight wingers are snipers, so the probability of the first winger selected being a sniper is

$$P(A) = \frac{3}{8}$$

If the first winger selected is a sniper, then there are seven remaining wingers to choose from, two of whom are snipers. Therefore,

$$P(B \mid A) = \frac{2}{7}$$

Applying the fundamental counting principle, the probability of randomly selecting two snipers for the farm team is the number of ways of selecting two snipers divided by the number of ways of selecting any two wingers.

$$\begin{aligned} P(A \text{ and } B) &= \frac{3 \times 2}{8 \times 7} \\ &= \frac{3}{28} \end{aligned}$$

There is a $\frac{3}{28}$ or 10.7% probability that two professional snipers will play for the farm team in the exhibition game.

Notice in Example 4 that, when two events A and B are dependent, you can still multiply probabilities to find the probability that they both happen. However, you must use the conditional probability for the second event. Thus, the probability that both events will occur is given by the **product rule for dependent events**:

$$P(A \text{ and } B) = P(A) \times P(B \mid A)$$

This reads as: "The probability that both A and B will occur equals the probability of A times the probability of B given that A has occurred."

Project Prep

When designing your game for the probability project, you may decide to include situations involving independent or dependent events. If so, you will need to apply the appropriate product rule in order to determine classical probabilities.

Example 5 Conditional Probability From Compound Probability

Serena's computer sometimes crashes while she is trying to use her e-mail program, OutTake. When OutTake "hangs" (stops responding to commands), Serena is usually able to close OutTake without a system crash. In a computer magazine, she reads that the probability of OutTake hanging in any 15-min period is 2.5%, while the chance of OutTake and the operating system failing together in any 15-min period is 1%. If OutTake is hanging, what is the probability that the operating system will crash?

Solution

Let event A be OutTake hanging, and event B be an operating system failure. Since event A can trigger event B, the two events are dependent. In fact, you need to find the conditional probability $P(B \mid A)$. The data from the magazine tells you that $P(A) = 2.5\%$, and $P(A \text{ and } B) = 1\%$. Therefore,

$$\begin{aligned} P(A \text{ and } B) &= P(A) \times P(B \mid A) \\ 1\% &= 2.5\% \times P(B \mid A) \\ P(B \mid A) &= \frac{1\%}{2.5\%} \\ &= 0.4 \end{aligned}$$

There is a 40% chance that the operating system will crash when OutTake is hanging.

Example 5 suggests a useful rearrangement of the product rule for dependent events.

$$P(B \mid A) = \frac{P(A \text{ and } B)}{P(A)}$$

This equation is sometimes used to define the conditional probability $P(B \mid A)$.

Key Concepts

- If A and B are independent events, then the probability of both occurring is given by $P(A \text{ and } B) = P(A) \times P(B)$.
- If event B is dependent on event A, then the conditional probability of B given A is $P(B \mid A)$. In this case, the probability of both events occurring is given by $P(A \text{ and } B) = P(A) \times P(B \mid A)$.

Communicate Your Understanding

1. Consider the probability of randomly drawing an ace from a standard deck of cards. Discuss whether or not successive trials of this experiment are independent or dependent events. Consider cases in which drawn cards are

a) replaced after each trial

b) not replaced after each trial

2. Suppose that for two particular events A and B, it is true that $P(B \mid A) = P(B)$. What does this imply about the two events? (*Hint*: Try substituting this equation into the product rule for dependent events.)

Practise

A

1. Classify each of the following as independent or dependent events.

	First Event	Second Event
a)	Attending a rock concert on Tuesday night	Passing a final examination the following Wednesday morning
b)	Eating chocolate	Winning at checkers
c)	Having blue eyes	Having poor hearing
d)	Attending an employee training session	Improving personal productivity
e)	Graduating from university	Running a marathon
f)	Going to a mall	Purchasing a new shirt

2. Amitesh estimates that he has a 70% chance of making the basketball team and a 20% chance of having failed his last geometry quiz. He defines a "really bad day" as one in which he gets cut from the team and fails his quiz. Assuming that Amitesh will receive both pieces of news tomorrow, how likely is it that he will have a really bad day?

3. In the popular dice game Yahtzee®, a Yahtzee occurs when five identical numbers turn up on a set of five standard dice. What is the probability of rolling a Yahtzee on one roll of the five dice?

Apply, Solve, Communicate

B

4. There are two tests for a particular antibody. Test A gives a correct result 95% of the time. Test B is accurate 89% of the time. If a patient is given both tests, find the probability that
   a) both tests give the correct result
   b) neither test gives the correct result
   c) at least one of the tests gives the correct result

5. **a)** Rocco and Biff are two koala bears participating in a series of animal behaviour tests. They each have 10 min to solve a maze. Rocco has an 85% probability of succeeding if he can smell the eucalyptus treat at the other end. He can smell the treat 60% of the time. Biff has a 70% chance of smelling the treat, but when he does, he can solve the maze only 75% of the time. Neither bear will try to solve the maze unless he smells the eucalyptus. Determine which koala bear is more likely to enjoy a tasty treat on any given trial.
   b) **Communication** Explain how you arrived at your conclusion.

6. Shy Tenzin's friends assure him that if he asks Mikala out on a date, there is an 85% chance that she will say yes. If there is a 60% chance that Tenzin will summon the courage to ask Mikala out to the dance next week, what are the odds that they will be seen at the dance together?

7. When Ume's hockey team uses a "rocket launch" breakout, she has a 55% likelihood of receiving a cross-ice pass while at full speed. When she receives such a pass, the probability of getting her slapshot away is $\frac{1}{3}$. Ume's slapshot scores 22% of the time. What is the probability of Ume scoring with her slapshot when her team tries a rocket launch?

8. **Inquiry/Problem Solving** Show that if A and B are dependent events, then the conditional probability $P(A \mid B)$ is given by

$$P(A \mid B) = \frac{P(A \text{ and } B)}{P(B)}.$$

9. A consultant's study found Megatran's call centre had a 5% chance of transferring a call about schedules to the lost articles department by mistake. The same study shows that, 1% of the time, customers calling for schedules have to wait on hold, only to discover that they have been mistakenly transferred to the lost articles department. What are the chances that a customer transferred to lost articles will be put on hold?

10. Pinder has examinations coming up in data management and biology. He estimates that his odds in favour of passing the data-management examination are 17:3 and his odds against passing the biology examination are 3:7. Assume these to be independent events.

   a) What is the probability that Pinder will pass both exams?

   b) What are the odds in favour of Pinder failing both exams?

   c) What factors could make these two events dependent?

11. **Inquiry/Problem Solving** How likely is it for a group of five friends to have the same birth month? State any assumptions you make for your calculation.

12. Determine the probability that a captured deer has the bald patch condition.

Chapter Problem

13. **Communication** Five different CD-ROM games, Garble, Trapster, Zoom!, Bungie, and Blast 'Em, are offered as a promotion by SugarRush cereals. One game is randomly included with each box of cereal.

   a) Determine the probability of getting all 5 games if 12 boxes are purchased.

   b) Explain the steps in your solution.

   c) Discuss any assumptions that you make in your analysis.

14. **Application** A critical circuit in a communication network relies on a set of eight identical relays. If any one of the relays fails, it will disrupt the entire network. The design engineer must ensure a 90% probability that the network will not fail over a five-year period. What is the maximum tolerable probability of failure for each relay?

C

15. a) Show that if a coin is tossed n times, the probability of tossing n heads is given by $P(A) = \left(\frac{1}{2}\right)^n$.

   b) What is the probability of getting at least one tail in seven tosses?

16. What is the probability of not throwing 7 or doubles for six consecutive throws with a pair of dice?

17. Laurie, an avid golfer, gives herself a 70% chance of breaking par (scoring less than 72 on a round of 18 holes) if the weather is calm, but only a 15% chance of breaking par on windy days. The weather forecast gives a 40% probability of high winds tomorrow. What is the likelihood that Laurie will break par tomorrow, assuming that she plays one round of golf?

18. **Application** The Tigers are leading the Storm one game to none in a best-of-five playoff series. After a playoff win, the probability of the Tigers winning the next game is 60%, while after a loss, their probability of winning the next game drops by 5%. The first team to win three games takes the series. Assume there are no ties. What is the probability of the Storm coming back to win the series?