PERMUTATIONS

Permutations

A permutation is an arrangement of a set of objects for which the order of the objects is important.

In how many ways can we arrange the letters of the word CAT?

CAT
CTA
ACT
ATC
TCA
TAC

There are six arrangements or permutations of the word CAT. By changing the order of the letters, you have a different permutation.

$\underline{3} \times \underline{2} \times \underline{1} = 6 = 3!$

Using the Fundamental Counting Principle, we would have 3 x 2 x 1 number of distinct arrangements or permutations.

In calculating permutations, we often encounter expressions such as 3 x 2 x 1. The product of consecutive natural numbers in decreasing order down to the number one can be represented using factorial notation:

3 x 2 x 1 = 3! This is read as "three factorial".

Evaluating Factorial Notation

10! = 10 x 9 x 8 x 7 x 6 x 5 x 4 x 3 x 2 x 1
= 3 628 800

By definition, for a natural number *n*:

$n! = n(n - 1)(n - 2)(n - 3) \times \ldots \times 3 \times 2 \times 1$

Simplify the following expressions.

a) $\frac{10!}{7!}$ b) $\frac{12!}{10!}$ c) $\frac{n!}{(n-1)!}$ d) $\frac{(n+3)!}{(n+1)!}$

a) $$\frac{10!}{7!} = \frac{10 \times 9 \times 8 \times \cancel{7 \times \ldots \times 3 \times 2 \times 1}}{\cancel{7 \times 6 \times 5 \times \ldots \times 3 \times 2 \times 1}}$$

$$= 10 \times 9 \times 8$$

$$= 720$$

b) $$\frac{12!}{10!} = \frac{12 \times 11 \times \cancel{10 \ldots \times 1}}{\cancel{10 \times 9 \times \ldots \times 1}}$$

$$= 12 \times 11$$

$$= 132$$

Evaluating Factorial Notation

c) $$\frac{n!}{(n-1)!} = \frac{n\cancel{(n-1)(n-2)\times\ldots\times3\times2\times1}}{\cancel{(n-1)(n-2)\times\ldots\times3\times2\times1}}$$

$= n$

d) $$\frac{(n+3)!}{(n+1)!} = \frac{(n+3)(n+2)\cancel{(n+1)\times\ldots3\times2\times1}}{\cancel{(n+1)(n)(n-1)\times\ldots\times3\times2\times1}}$$

$= (n+3)(n+2)$
$= n^2 + 5n + 6$

Express as a single factorial.

$5\times4\times3\times2\times1$

a) $9 \times 8 \times 7 \times 6 \times 5!$ $= 9!$

b) $n(n-1)\boxed{(n-2)!}$ $= n!$

c) $(n-r+1)(n-r)(n-r-1)!$ $= (n-r+1)!$ *

Finding the Number of Permutations

1. Determine the number of ways that seven boys can line up.

   $7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040$ $\underline{7}\ \underline{6}\ \underline{5}\ \underline{4}\ \underline{3}\ \underline{2}\ \underline{1}$

   There are 5040 different ways that seven boys can line up.

   $= 7!$

> **The number of permutations of *n* different objects taken all at a time is *n*!**

2. In how many ways can the letters of the word **HARMONY** be arranged?

   $7! = 5040$

   There are 5040 ways of arranging the seven letters.

Finding the Number of Permutations

$8! \rightarrow$ all letters are used

How many three-letter permutations can be formed from the letters of the word DINOSAUR?

only 3 $\rightarrow$ $\underline{8}\ \underline{7}\ \underline{6}$

$8 \times 7 \times 6 = {}_8P_3$

$$\frac{8}{1^{st}} \times \frac{7}{2^{nd}} \times \frac{6}{3^{rd}}$$

There are 336 ways.

336 represents the number of permutations of eight objects taken three at a time.

This may be written as ${}_8P_3$. (This is read as "eight permute three".)

In general, if we have n objects but only want to select r objects at a time, the number of different arrangements is:

$$\boxed{{}_nP_r = \frac{n!}{(n-r)!}}$$

$${}_8P_3 = \frac{8!}{(8-3)!} = \frac{8!}{5!}$$

The number of permutations of n objects taken n at a time is:

$$\boxed{{}_nP_n = n!}$$

$${}_nP_n = \frac{n!}{(n-n)!} = \frac{n!}{0!} = \frac{n!}{1} = n!$$

Finding the Number of Permutations

A special case of this formula occurs when $n = r$.

$${}_3P_3 = 3!$$

$${}_3P_3 = \frac{3!}{(3-3)!}$$

$${}_3P_3 = \frac{3!}{0!}$$

From these two results, we can see that $0! = 1$.
To have meaning when $r = n$, we define $0!$ as 1.

1. Find the value of each expression:

   a) ${}_6P_3 = 120$ b) ${}_{10}P_7 = 604\,800$

2. Using the letters of the word PRODUCT, how many four-letter arrangements can be made?

   ${}_7P_4 = 840$ There are 840 arrangements.

Finding the Number of Permutations

Solve for *n*. $_nP_2 = 90$

$$\frac{n!}{(n-2)!} = 90$$

$$\frac{n(n-1)\cancel{(n-2)\times\ldots\times 3\times 2\times 1}}{\cancel{(n-2)(n-3)\times\ldots 3\times 2\times 1}} = 90$$

$$n(n-1) = 90$$

$$n^2 - n = 90$$

$$n^2 - n - 90 = 0$$

$$(n-10)(n+9) = 0$$

$n - 10 = 0$ OR $n + 9 = 0$ $\quad n \in \mathbb{N}$

$n = 10$ $\quad n = -9$

$\boxed{\text{Therefore, } n = 10.}$

Finding the Number of Permutations

1. How many six-letter words can be formed from the letters of TRAVEL? (Note that letters cannot be repeated)

a) If any of the six letters can be used:

$${}_6P_6 = 6!$$
$$= 720$$

b) If the first letter must be "L":

L is the only choice

$$1 \times {}_5P_5 = 1 \times 5!$$
$$= 120$$

c) If the second and fourth letters are vowels:

$$2 \times 1 \times 4! = 48$$

4 2 3 1 2 1

V V

1st vowel 2 choices

2nd vowel 1 choice

d) If the "A" and the "V" must be adjacent:

(Treat the AV as one letter - this grouping can be arranged 2! ways.)

$$5! \times 2! = 240$$

2 choices AV or VA

AV or VA

$5! + 5! = 5! \times 2!$

or

Finding the Number of Permutations

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1 _ _ _ _ _

$${}_5P_5$$

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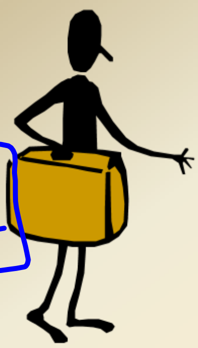

c) If the second and fourth letters are vowels:

$$2 \times 1 \times 4! = 48$$

$4! = {}_4P_4$

_ $\underset{V}{2}$ _ $\underset{V}{1}$ _ _

d) If the "A" and the "V" must be adjacent:

(Treat the AV as one letter - this grouping can be arranged 2! ways.)

$$5! \times 2! = 240$$

$\underbrace{A_1+A_2+A_3+A_4+A_5}_{5!}$ $\underline{8}$ $\underline{7}$ $\underline{6}$ $\underline{5}$ $\underline{4}$ $\underline{3}$ $\underline{2}$ $\underline{1}$ $\times\ 5!$

5! One Choice

arrange Algebra books

Finding the Number of Permutations

2. **A bookshelf contains five different algebra books and seven different physics books. How many different ways can these books be arranged if the algebra books are to be kept together?**

For the five algebra books: 5!

The five algebra books are considered as one item, therefore, you have eight items (7 + 1) to arrange: 8!

Total number of arrangements = 8! x 5!

= 40 320 x 120

= 4 838 400

3. **In how many ways can six boys and six girls be arranged on a bench, if no two people of the same gender can sit together?**

Case 1: b g b g b g b g b g b g

+

Case 2: g b g b g b g b g b g b

Boys: 6!

Girls: 6!

The total number of arrangements is 6! x 6! x 2 = 1 036 800.

Finding the Number of Permutations

2. A bookshelf contains five different algebra books and seven different physics books. How many different ways can these books be arranged if the algebra books are to be kept together?

For the five algebra books: 5! ${}_5P_5$

The five algebra books are considered as one item, therefore, you have eight items (7 + 1) to arrange: 8!

${}_8P_8 = 8!$

Total number of arrangements = 8! x 5!

= 40 320 x 120

= 4 838 400

3. In how many ways can six boys and six girls be arranged on a bench, if no two people of the same gender can sit together?

b	g	b	g	b	g	b	g	b	g	b	g
g	b	g	b	g	b	g	b	g	b	g	b

Boys: 6!
Girls: 6!

The total number of arrangements is 6! x 6! x 2 = 1 036 800.

Finding the Number of Permutations

4. In how many ways can the letters of the word *MATHPOWER* be arranged if: 9 different letters → 9!

   a) there are no further restrictions?

   9! = 362 880

   b) the first letter must be a P and the last letter an A? 7! x 2!

   1 x 7! x 1 = 5040

   c) the letters MATH must be together? OOOO 4! | $\underline{6}\ \underline{5}\ \underline{4}\ \underline{3}\ \underline{2}\ \underline{1}$ MATH as 1 choice

   6! x 4! = 17 280 → # of ways to arrange MATH

   d) the letters MATH must be together and in that order?

   1 x 6! = 720

5. How many numbers can be made from the digits 2, 3, 4, and 5, if no digit can be repeated? not just 4-digit numbers

   235, 243, ... 3 digits 1 digit 2, 3, 4, 5

   $4! + {}_4P_3 + {}_4P_2 + {}_4P_1 = 64$

   4-digits 2345, 2534, ...

   2 digits 23, 24, 25, ...

Finding the Number of Permutations

$M_3INIM_2UM_3$
$M_3INIM_1UM_2$
$M_2INIM_2UM_3$
$M_1INIM_3UM_2$

6. How many arrangements of four letters are there from the word PREACHING?

$${}_9P_4 = 3024$$

7. How many distinct arrangements of MINIMUM are there?

$$\textit{number of arrangements} = \frac{7!}{3!2}$$

$$= 420$$

$M_1INIM_2UM_3$
$M_2INIM_3UM_2$

8. There are six different flags available for signaling. A signal consists of at least four flags tied one above the other. How many different signals can be made?

$6! = {}_6P_6$

$${}_6P_4 + {}_6P_5 + {}_6P_6 = 1800$$

$$\frac{6!}{2!} + \frac{6!}{1!} + \frac{6!}{0!} = 360 + 720 + 720$$

Solving Equations Involving Permutations

Solve ${}_nP_2 = 30$ algebraically.

$$\frac{n!}{(n-2)!} = 30$$

$$\frac{n(n-1)(n-2)!}{(n-2)!} = 30$$

$$n(n-1) = 30$$

$$n^2 - n = 30$$

$$n^2 - n - 30 = 0$$

$$(n-6)(n+5) = 0$$

$$n = 6 \text{ or } n = -5$$

Therefore, $n = 6$.

Permutations with Repetition (Identical items)

How many four-letter words can be made using the letters of PEER?

$_4P_4 = 4!$

Note that PEER and PEER are not distinguishable, therefore, they count as one permutation.
When this happens, you must divide by the factorial of the number of repeated terms.

For PEER, the number of arrangements is:

$$\frac{4!}{2!} = 12 \text{ arrangement}$$

The number of permutations of n objects taken n at a time, if there are a alike of one kind, and b alike of another kind, c alike of a third kind, and so on, is:

$$\frac{n!}{a!b!c!\ldots}$$

Permutations with Restrictions

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1. In how many ways can the letters of the word ENGINEERING be arranged?

3Es 3Ns 2Is 2Gs

$$\text{number of arrangements} = \frac{11!}{3!3!2!2!}$$

$$= 277\ 200$$

2. Naval signals are made by arranging coloured flags in a vertical line. How many signals can be made using six flags, if you have:
   a) 3 green, 1 red, and 2 blue flags?
   b) 2 red, 2 green, and 2 blue flags?

(a) $\frac{6!}{3!2!}$ (b) $\frac{6!}{2!2!2!}$

(a) $\text{number of arrangements} = \frac{6!}{3!2!}$

$= 60$

(b) $\text{number of arrangements} = \frac{6!}{2!2!2!}$

$= 90$

3 6 7 6 5 4 3 2 1 3 2 6 6 5 4 3 2 1

Permutations with Restrictions

2 T's 3 I's.

3. Find the number of arrangements of the letters of UTILITIES:

a) if each begins with one I and the second letter is not an I.

3 x 6 x 7! Total

$$\text{arrangements} = \frac{3 \times 6 \times 7!}{3!2!}$$

$= 7560$

b) if each begins with exactly $\boxed{\text{two I's.}}$ can't have 3 I's.

3 x 2 x 6 x 6! Total

$$\text{arrangements} = \frac{3 \times 2 \times 6 \times 6!}{3!2!}$$

$= 2160$

C H R

4. How many arrangements are there, using all the letters of the word REACH, if the consonants must be in alphabetical order?

HEARC CEAHR

If the order of letters cannot be changed, then treat these letters as if they were identical.

$$\text{arrangements} = \frac{5!}{3!}$$

$= 20$

MDM4U **Permutations with Identical Items** Date: ___________

How many ways can we arrange the letters from MOO?

MO_1O_2 MOO OMO OOM $\frac{3!}{2!}$

MO_2O_1

There are 3 distinct words.

How many ways can we arrange the letters from MOOO?

$MO_1O_2O_3$ $\frac{4!}{3!}$

$MO_1O_3O_2$

$MO_2O_1O_3$

$MO_2O_3O_1$

$MO_3O_1O_2$

$MO_3O_2O_1$

There are only 4 different words.

Now consider the letter MOOM

$M_1M_2O_1O_2$

$M_1M_2O_2O_1$

$M_2M_1O_1O_2$

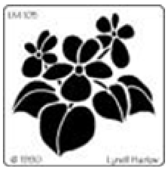
$M_2M_1O_2O_1$

2! 2!

$$\frac{4!}{2!2!} = 6$$

Example 1:

Determine the number of arrangements of 3 violets and 2 roses in a block vase.



n = 5 a = 3 b = 2

$$\frac{5!}{3!\,2!}$$

Example 2:

In how many ways could 5 pennies, 3 nickels, 2 dimes and a quarter be arranged in a line?

n = a = b = c = d =

$$\frac{11!}{5!\,3!\,2!}$$

Example 3:

a) How many arrangements are there of the letters in the word MATHEMATICS?

b) How many ways of these arrangements begin with the letter M?

c) How many of the arrangements in part a) would have the T's together?

Example 3:

11

a) How many arrangements are there of the letters in the word MATHEMATICS?

2Ms 2Ts 2As

$$\frac{11!}{2!2!2!} = 4989600$$

b) How many ways of these arrangements begin with the letter M?

$\underline{2}\ \underline{10}\ \underline{9}\ \underline{8}\ \underline{7}\ \underline{6}\ \underline{5}\ \underline{4}\ \underline{3}\ \underline{2}\ \underline{1}$

$$\frac{2 \times 10!}{2!2!2!} = 907200$$

$$\frac{\cancel{2}}{\cancel{2}!} \times \boxed{\frac{10!}{2!2!}}$$

c) How many of the arrangements in part a) would have the T's together?

$\boxed{TT}$ as one choice

$\underbrace{\underline{T}\ \underline{T}}\ \underline{9}\ \underline{8}\ \underline{7}\ \underline{6}\ \underline{5}\ \underline{4}\ \underline{3}\ \underline{2}\ \underline{1}$

↓

10 different locations

$$10 \times \frac{9!}{2!2!} = 907200$$