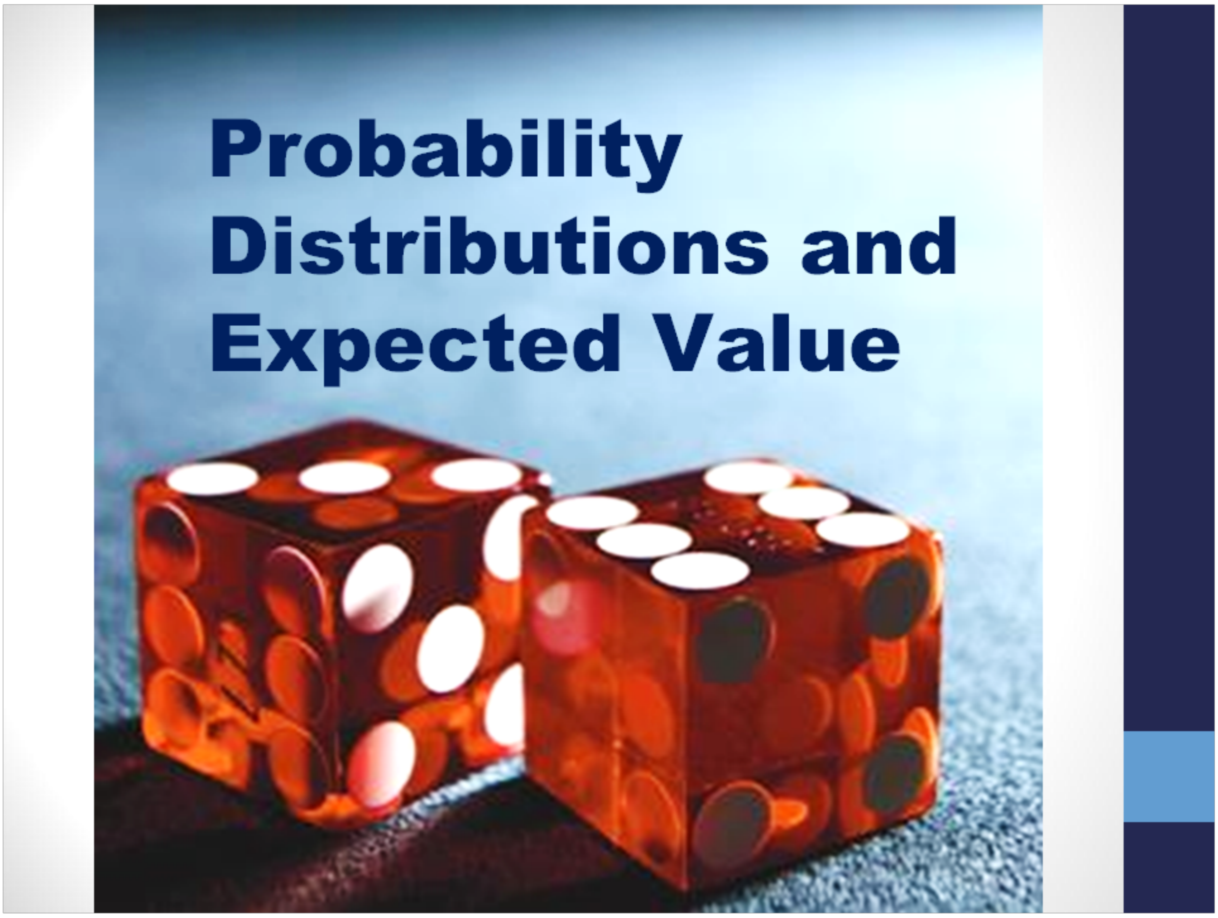

Probability
Distributions and
Expected Value

Probability Distributions

- Probability distributions show the probabilities of all the possible outcomes of an experiment, instead of just a particular or individual outcome
- Many probability experiments have numerical outcomes which can be counted or measured

Random Variable

- A random variable, X, has a single value *x* for each outcome in an experiment
- Ex. If 'X' is the number rolled with a die, then *x* has a different value for each of the 6 possible outcomes
- Random variables can be **discrete** or **continuous**
- To show all the possible outcomes, a chart is normally used

Discrete Random Variable

- Discrete random variables take values that are separate from each other (or that can be "counted")
- The number of possible values can be small finite

Continuous Random Variable

- Continuous random variables have an "infinite" number of possible values in a continuous interval (or measurements that can have an unlimited decimal place)

Discrete vs. Continuous Variable

Classify each as either discrete or continuous.

- number of customers on a newspaper route
- distance from here to Eglinton/Yonge Station
- amount of money that you can make in a job

Ask yourself if the values can be measured to an infinite point (Does it require integers or fractions of a decimal?)

Discrete vs. Continuous Variable

Classify each as either discrete or continuous.

- number of customers on a newspaper route
  (Discrete)
- distance from here to Eglinton/Yonge Station
  (Continuous)
- amount of money that you can make in a job
  (Continuous)

Do the values require integers or fractions of a decimal?

Uniform Probability Distributions

Let a Discrete Random Variable (X) be the possible outcomes when rolling a die.

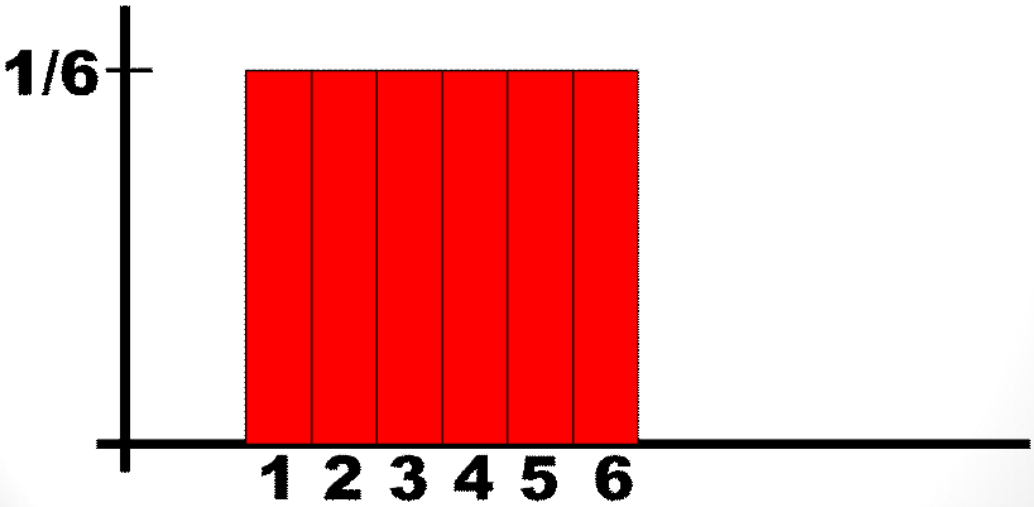
The probability distribution could be written in a table:

X	1	2	3	4	5	6
P(X) = x	1/6	1/6	1/6	1/6	1/6	1/6

This is a uniform distribution since all the probabilities for the outcomes are the same.

Uniform Probability Distributions

A graph would look like this…



Uniform Probability Distributions

- All outcomes in a uniform probability distribution are equally likely in any single trial
- The sum of the probabilities in a uniform probability distribution is 1

Note:

All probability distributions must sum to 1 since they include all possible outcomes.

Uniform Probability Distributions

For project presentation, students in a math class are divided into 5 groups, A to E. If the order in which the groups make their presentation is randomly selected, determine the probability distribution for the order of the group presentation with Group A presenting their project first, second, third, fourth or fifth.

Uniform Probability Distributions

Since each group has an equal probability for choosing each of the five positions, each probability is 1/5.

Random Variable, x	Probability, P(x)
Position 1	1/5
Position 2	1/5
Position 3	1/5
Position 4	1/5
Position 5	1/5
Position 6	1/5

Uniform Probability Distributions

This distribution has a uniform probability distribution.

The **probability** in a discrete uniform distribution:

For all values of *x*,

P(X) = 1/n

where n is the number of possible outcomes in the experiment

What about when tossing a coin?

n=? P(x)=?

Expected Value

- An Expectation or Expected Value, E(X), is the predicted average (or weighted mean) of all possible outcomes of a probability experiment
- The expectation is equal to the sum of the products of each outcome with its probability i.e. (random variable = x) × P(x)

$$E(X) = x_1 P(X = x_1) + x_2 P(X = x_2) + \ldots + x_n P(X = x_n)$$

$$= \sum_{i=1}^{n} x_i P(X = x_i)$$

Expected Value

The capital sigma, $\sum$, means "the sum of".

$$E(X) = x_1 P(X = x_1) + x_2 P(X = x_2) + \ldots + x_n P(X = x_n)$$
$$= \sum_{i=1}^{n} x_i P(X = x_i)$$

The sum is from the first term ($i = 1$) to the nth term ($i = n$).

i.e. the sum of the terms in the form (x)(P[x])

Expected Value of a Fair Game

Consider a game of rolling a single die. If you roll an even number, you gain that number of points. If you roll an odd number, you lose that number of points. The probability distribution of points in this game:

Number on Upper Face	Points, x	Probability, $P(x)$
1	−1	$\frac{1}{6}$
2	2	$\frac{1}{6}$
3	−3	$\frac{1}{6}$
4	4	$\frac{1}{6}$
5	−5	$\frac{1}{6}$
6	6	$\frac{1}{6}$

Expected Value of a Fair Game

Number on Upper Face	Points, x	Probability, $P(x)$
1	−1	$\frac{1}{6}$
2	2	$\frac{1}{6}$
3	−3	$\frac{1}{6}$
4	4	$\frac{1}{6}$
5	−5	$\frac{1}{6}$
6	6	$\frac{1}{6}$

What is the expected number of points per roll?

$$E(x)=(-1)\left(\tfrac{1}{6}\right)+(2)\left(\tfrac{1}{6}\right)+(-3)\left(\tfrac{1}{6}\right)+(4)\left(\tfrac{1}{6}\right)+(-5)\left(\tfrac{1}{6}\right)+(6)\left(\tfrac{1}{6}\right)$$

$$=\tfrac{1}{6}(-1+2-3+4-5+6)$$

$$=\tfrac{1}{6}(3)=0.5$$

Expected Value of a Fair Game

Number on Upper Face	Points, x	Probability, $P(x)$
1	−1	$\frac{1}{6}$
2	2	$\frac{1}{6}$
3	−3	$\frac{1}{6}$
4	4	$\frac{1}{6}$
5	−5	$\frac{1}{6}$
6	6	$\frac{1}{6}$

What is the expected number of points per roll?

$E(X) = 0.5$

Expected Value of a Fair Game

Number on Upper Face	Points, x	Probability, $P(x)$
1	−1	$\frac{1}{6}$
2	2	$\frac{1}{6}$
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6	6	$\frac{1}{6}$

Is this a fair game? Why or Why Not?

If it's a fair game, no advantage to win or disadvantage to lose → $E(x)=0$

Expected Value of a Fair Game

Number on Upper Face	Points, x	Probability, $P(x)$
1	−1	$\frac{1}{6}$
2	2	$\frac{1}{6}$
3	−3	$\frac{1}{6}$
4	4	$\frac{1}{6}$
5	−5	$\frac{1}{6}$
6	6	$\frac{1}{6}$

Is this a fair game? Why or Why Not? **No!**

* **For a game to be fair, the expected outcome must be 0.**

Expected Value

A building store has different sizes of ceiling boards. There are six 6-m boards, five 8-m boards, three 4-m boards and four 10-m boards.

Sum = 18

Length of board (M), x	Probability, P(x)
4-m	3/18
6-m	6/18
8-m	5/18
10-m	4/18

What is the expected length of these boards?

$$E(x) = 4\left(\frac{3}{18}\right) + 6\left(\frac{6}{18}\right) + 8\left(\frac{5}{18}\right) + 10\left(\frac{4}{18}\right) = 7\frac{1}{9}$$

$$= 7.11\text{m}$$

Expected Value

Length of board (M), x	Probability, $P(x)$
4-m	3/18
6-m	6/18
8-m	5/18
10-m	4/18

What is the expected length of the ceiling boards?

$E(x) = 4(3/18) + 6(6/18) + 8(5/18) + 10(4/18)$

The expected length is 7.11 m.

Example 1

A game is defined by rules that two dice are rolled and the player wins varying amounts depending on the sum of the two dice in accordance with the following table.

SUM	2	3	4	5	6	7	8	9	10	11	12
Payout	$10	$9	$8	$7	$6	$5	$6	$7	$8	$9	$10

What should a player expect to win?

What would be a fair value to play?

Example 1

SUM	2	3	4	5	6	7	8	9	10	11	12
Payout	\$10	\$9	\$8	\$7	\$6	\$5	\$6	\$7	\$8	\$9	\$10

What should a player expect to win?

$$E(\$) = \$10\left(\frac{1}{36}\right) + \$9\left(\frac{2}{36}\right) + \$8\left(\frac{3}{36}\right) + \$7\left(\frac{4}{36}\right) + \$6\left(\frac{5}{36}\right) + \$5\left(\frac{6}{36}\right) + \$6\left(\frac{5}{36}\right) + \$7\left(\frac{4}{36}\right) + \$8\left(\frac{3}{36}\right) + \$9\left(\frac{2}{36}\right) + \$10\left(\frac{1}{36}\right)$$

$$= \frac{\$10 + \$18 + \$24 + \$28 + \$30 + \$30 + \$30 + \$28 + \$24 + \$18 + \$10}{36} = \frac{\$250}{36} = \$6.94$$

Expected payout is \$6.94

What would be a fair value to play?

"Fair" value to play is \$6.94

Example 2

Three people are chosen from a group consisting of 4 men and 3 women.

- Determine the probability of the chosen committee having at least one woman on it.
- Determine the expected value of choosing a woman.

Example 2

Three people are chosen from a group consisting of 4 men and 3 women. Determine the probability of the chosen committee having at least one woman on it.

$$W = \text{number of women selected}$$
$$W = \{0, 1, 2, 3\}$$
$$P(W \geq 1) = 1 - P(\text{no women})$$
$$= 1 - \frac{\binom{3}{0}\binom{4}{3}}{\binom{7}{3}}$$
$$= 1 - \frac{1 \times 4}{35}$$
$$= \frac{31}{35}$$

Example 2

Three people are chosen from a group consisting of 4 men and 3 women. Determine the expected value of choosing a woman.

W	0	1	2	3
P(W)	$=\frac{\binom{3}{0}\binom{4}{3}}{\binom{7}{3}}$ $=\frac{4}{35}$	$=\frac{\binom{3}{1}\binom{4}{2}}{\binom{7}{3}}$ $=\frac{18}{35}$	$=\frac{\binom{3}{2}\binom{4}{1}}{\binom{7}{3}}$ $=\frac{12}{35}$	$=\frac{\binom{3}{3}\binom{4}{0}}{\binom{7}{3}}$ $=\frac{1}{35}$

$$E(W)=0\left(\frac{4}{35}\right)+1\left(\frac{18}{35}\right)+2\left(\frac{12}{35}\right)+3\left(\frac{1}{35}\right)$$
$$\doteq 1.3$$